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# CONJUGACY CLASSES IN A $p$-GROUP OF MAXIMAL CLASS OF ORDER $p^8$ *

by

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In this paper, $G$ will denote a $p$-group of maximal class, i.e., a finite group of order $p^m$ and nilpotency class $m-1$, where $p$ is a prime number We complete the lower central series of $G$, $G > Y_2(G) > \cdots > Y_m(G) = 1$, with a maximal subgroup of $G$, $Y_1(G)$, defined by the relation $Y_1(G)/Y_4(G) = C_{G/Y_4(G)}(Y_2(G)/Y_4(G))$. We will write $Y_i$ instead of $Y_i(G)$ when there is no possible confusion. We will also denote by $c = c(G)$ the degree of conmutativity of $G$. If we take an element $s \in G - (Y_1 \cup C_G(Y_{m-2}))$, another element $s_1 \in Y_1 - Y_2$ and define recursively $s_i = [s_{i-1}, s]$ for $i = 2, \ldots, m-1$, we call the tuple $(s, s_1, \ldots, s_{m-1})$ a generator $G$-system and the tuple

$$\sigma_G = \left(r_G(sY_1), r_{Y_1}(s_1Y_2), \ldots, r_{Y_i}(s_iY_{i+1}), \ldots, r_{Y_{m-1}}(s_{m-1}Y_m)\right)$$

a numerical $G$-system. Clearly, $\sigma_G$ is an invariant of $G$.

We are mostly concerned with the determination of $r(G)$ and $\sigma_G$ for a $p$-group of maximal class. This problem was firstly considered in the paper [2], where the fundamentals of this theory are developed. There can be found complete lists for these invariants for $G$ metabelian and $|G| \leq p^7$. In this paper we obtain $r(G)$ and $\sigma_G$ for a $p$-group of maximal class and order $p^8$. Before listing the different possibilities which appear in this case, we give some definitions and state a useful lemma.

DEFINITION. Let $G$ be a $p$-group of maximal class with $Y_3$ abelian and let $i \in \{3, \ldots, m - c(G) - 2\}$. We say that $G$ has a **jump** at $Y_i$ if $[Y_1, Y_i] = Y_{i+c+2}$.

LEMMA. Let $G$ be a $p$-group of maximal class with $Y_3$ abelian and $Y_2' \leq Z(Y_2)$. Suppose, in addition, that $G$ has exactly two jumps, at $Y_u$ and $Y_{m-c-2}$. Then, there exist elements $t_i \in Y_i - Y_{i+1}$ $(i = 1, 2, \ldots, m-1)$ such that

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$$[t_i, t_1] = \begin{cases} t_{i+c+1}, & \text{if } 3 \le i \le m-c-3 \text{ and } i \ne u, \\ t_{u+c+2}^{a_{u+c+2}} \cdots t_{m-1}^{a_{m-1}}, & \text{with } a_j \in \{0, \dots, p-1\} \ \forall j, \text{ if } i = u, \\ 1, & \text{if } i \ge m-c-2. \end{cases}$$

$$[t_i, t_2] = \begin{cases} t_{i+c+2}, & \text{if } 3 \le i \le m-c-3, \\ 1, & \text{if } i \ge m-c-2. \end{cases}$$

$$[t_2, t_1] = \begin{cases} t_{c+3}, & \text{if } u \ge 4, \\ t_{c+4}, & \text{if } u = 3 \text{ and } [Y_1, Y_1] = Y_{c+4}, \\ 1, & \text{if } u = 3 \text{ and } [Y_1, Y_1] = Y_{c+5}. \end{cases}$$

DEFINITION. If $G$ is a $p$-group of maximal class which satisfies the conditions of the preceding lemma, the equation with coefficients in $\mathbf{Z}/p\mathbf{Z}$

$$x^{m-u-c-2} - a_{u+c+2}x^{m-u-c-3} - \cdots - a_{m-2}x - a_{m-1} = 0$$

is called the **fundamental equation** associated to the group $G$. If $[Y_1, Y_1] = Y_{c+2+j}$ ($j \in \{1,2,3\}$) and we denote by $S = S(a_{u+c+2}, \dots, a_{m-1})$ the number of different solutions in $\mathbf{Z}/p\mathbf{Z}$ of the fundamental equation, we say that $G$ is a **$\Gamma_{i,S}(m, c; u)$-group**.

In the following theorem we denote $\overline{G} = G/Z(G)$ and $\overline{G}_{(2)} = G/Z_2(G) = G/Y_{m-2}(G)$. We also write $\tau_n = (p^n, p^{n-1}, \dots, p, 1)$ for $n \ge 0$.

THEOREM. Let $G$ be a $p$-group of maximal class and order $p^8$. Set $r(G) = f_k(|G|)$. Then,

$$k \in \{0, 1, 2, 3, p+1, p+2, p+3, 2p+2, p^2+p+2, p^2+2p+2, p^3+p^2+2p+2\}$$

and one of the following conditions holds:

A) $G$ is metabelian
  1) $c(G) = 6$, $\sigma_G = (p, \tau_6)$ and $k = p^3 + p^2 + 2p + 2$.
  2) $c(G) = 4$, $\sigma_G = (p, p^5, \tau_5)$ and $k = p^2 + 2p + 2$.
  3) $c(G) = 3$, $\sigma_G = (p, p^4, \tau_5)$ and $k = 2p + 2$.
  4) $c(G) = 2$, $\sigma_G = (p, p^3, \tau_5)$ and $k = p + 2$.
  5) $c(G) = 1$, $\sigma_G = (p, p^2, \tau_5)$ and $k = p + 1$.

B) $G$ is not metabelian and $\overline{G}$ is metabelian
  1) $c(G) = 2$, $c(\overline{G}) = 5$, $\sigma_G = (p, p^5, p^4, \tau_4)$ and $k = p^2 + p + 2$.
  2) $c(G) = 2$, $c(\overline{G}) = 3$, $\sigma_G = (p, p^4, p^4, \tau_4)$ and $k = p + 2$.
  3) $c(G) = c(\overline{G}) = 2$, $[Y_1, Y_4] = 1$, $\sigma_G = (p, p^4, p^4, \tau_4)$ and $k = p + 2$.
  4) $c(G) = c(\overline{G}) = 2$, $[Y_1, Y_4] \ne 1$, $\sigma_G = (p, p^3, p^4, \tau_4)$ and $k = 2$.
  5) $c(G) = c(\overline{G}) = 1$, $\sigma_G = (p, p^2, p^4, \tau_4)$ and $k = 1$.
  6) $c(G) = 0$, $c(\overline{G}) = 5$, $\sigma_G = (2p-1, p^5, p^4, p^3, \tau_3)$ and $k = p^2 + p + 2$.
  7) $c(G) = 0$, $c(\overline{G}) = 3$, $\sigma_G = (2p-1, p^4, p^4, p^3, \tau_3)$ and $k = p + 2$.
  8) $c(G) = 0$, $c(\overline{G}) = 2$, $\sigma_G = (2p-1, p^3, p^4, p^3, \tau_3)$ and $k = 2$.
  9) $c(G) = 0$, $c(\overline{G}) = 1$, $\sigma_G = (2p-1, p^2, p^4, p^3, \tau_3)$ and $k = 1$.

C) $G$ and $\overline{G}$ are not metabelian

1) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] = 1$, $G$ is a $\Gamma_{3,2}(8,1;3)$-group, $\sigma_G = (p, p^4 + 2p^3 - 2p^2, p^3, \tau_4)$ and $k = p + 3$.

2) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] = 1$, $G$ is a $\Gamma_{3,1}(8,1;3)$-group, $\sigma_G = (p, p^4 + p^3 - p^2, p^3, \tau_4)$ and $k = p + 2$.

3) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] = 1$, $G$ is a $\Gamma_{3,0}(8,1;3)$-group, $\sigma_G = (p, p^4, p^3, \tau_4)$ and $k = p + 1$.

4) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] = 1$, $G$ is a $\Gamma_{2,2}(8,1;3)$-group, $\sigma_G = (p, 3p^3 - 2p^2, p^3, \tau_4)$ and $k = 3$.

5) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] = 1$, $G$ is a $\Gamma_{2,1}(8,1;3)$-group, $\sigma_G = (p, 2p^3 - p^2, p^3, \tau_4)$ and $k = 2$.

6) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] = 1$, $G$ is a $\Gamma_{2,0}(8,1;3)$-group, $\sigma_G = (p, p^3, p^3, \tau_4)$ and $k = 1$.

7) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] = 1$ is the only jump of $G$, $\sigma_G = (p, p^3, p^3, \tau_4)$ and $k = 1$.

8) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] \neq 1$, $\sigma_G = (p, p^4, p^3, \tau_4)$ and $k = p + 1$.

9) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] \neq 1$, $\sigma_G = (p, p^3, p^3, \tau_4)$ and $k = 1$.

10) $c(G) = c(\overline{G}) = 1$, $[Y_1, Y_5] \neq 1$, $\sigma_G = (p, p^2, p^3, \tau_4)$ and $k = 0$.

11) $c(G) = 0$, $c(\overline{G}) = 1$, $c(\overline{G}_{(2)}) = 4$, $\sigma_G = (2p - 1, p^4, p^3, p^3, \tau_3)$ and $k = p + 1$.

12) $c(G) = 0$, $c(\overline{G}) = 1$, $c(\overline{G}_{(2)}) = 2$, $\sigma_G = (2p - 1, p^3, p^3, p^3, \tau_3)$ and $k = 1$.

13) $c(G) = 0$, $c(\overline{G}) = c(\overline{G}_{(2)}) = 1$, $\sigma_G = (2p - 1, p^3, p^3, p^3, \tau_3)$ and $k = 1$.

14) $c(G) = 0$, $c(\overline{G}) = c(\overline{G}_{(2)}) = 1$, $\sigma_G = (2p - 1, p^2, p^3, p^3, \tau_3)$ and $k = 0$.

## REFERENCES


[1] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45-92.

[2] A. Vera-López and B. Larrea, *On p-groups of maximal class*, to appear in J. Algebra.