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On the Structure of G-spaces

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In this paper we study the class of those locally convex spaces which are reduced projective limits of Banach spaces with approximable linking maps, named **G**-spaces. This study is carried out from the point of view of approximation structures in Schwartz spaces.

Obviously each nuclear space is a **G**-space and each **G**-space is a Schwartz space with the Approximation Property. The first implication is strict, and it is an open question (posed by Ramanujan in | 6 |) whether the second is so or not. It has been proved by Nelimarkka in | 5 | that each Fréchet-Schwartz space with the Bounded Approximation Property is a **G**-space.

In this paper we introduce two "local" versions of the BAP with respect to a finite number of seminorms (property G) and with respect to bounded sets (property L). These properties characterize the Schwartz G-spaces as precisely the **G**-spaces, and the co-Schwartz L-spaces as the spaces whose strong dual is a **G**-space. From this it follows that each Schwartz space with the BAP is a **G**-space.

There is another sense in which it could be said that an lcs has locally the BAP: when it possesses a fundamental system of neighborhoods of zero such that the associated Banach spaces have the BAP. Let us call the Schwartz spaces with this property **G***-spaces. The question of whether **G** and **G***-spaces coincide arises. It is a generalization of the following problem of Schottenloher | 7 |:

An lcs E is said to be a DFC-space if $E=F'_c$ for some Fréchet space F (here F'_c represents the dual space of F endowed with the topology of the convergence over the compact sets of F). The question of Schottenloher was:

"Does every DFC-space with AP have a fundamental system of neighborhoods of zero such that the associated Banach spaces have the BAP ? "

Proposition. Let F be a Fréchet-Montel space. Then they are equivalent: 1. F has AP 2. F is an L-space 3. F'_b is a G-space 4. F'_b is a G-space 5. F'_b has AP.

from where it follows that we can remove the assumption "Montel" on F to obtain the results for F'_c instead of F'_b.

Therefore the problem of Schottenloher can be viewed as the problem: "Is each **G**-space a **G***-space?" for some special **G**-spaces.

In | 4 | M.L.Lourenço proves:
"If E is a DFC-space with AP, then E is a compact projective limit of a family of Banach spaces with a monotone Schauder basis".
this is tantamount to saying that special **G**-spaces (DFC-spaces with AP) are subspaces of special **G***-spaces (with a local monotone basis). In the paper under consideration we prove:

Theorem. A Hausdorff lcs E is a G-space if and only if it is a locally complemented subspace of a G*-space.

Thus, the result of Lourenço gives, for those particular **G**-spaces, better information about the "big" space, while our theorem gives a better knowledge of the quality of the embedding, valid for all spaces.

After a combimation of our results, we se that the class of **G**-spaces can be regarded as a generalization of the class of Schwartz spaces with BAP: firstly we proved that Schwartz spaces with local BAP are the **G**-spaces. On the other hand, Schwartz spaces with local FDD are **G***-spaces. In this way, the above theorem generalizes (adding "local") before the key words) a well-known structure theorem of Benndorf | 1 |:

" E is a Fréchet Schwartz space with the BAP if and only if it is a complemented subspace of a Fréchet Schwartz space with an FDD". The passage to "local" allows us to remove the metrizabiity condition.

It is perhaps worth mentioning that another purely internal description of the **G**-spaces was obtained in |2| in terms of the so-called Uniform Approximation Property (UAP). This property has some relevance in studying approximation structures in Schwartz and co-Schwartz spaces (see |3|).

References

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