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FIXED POINTS FOR COMPOSITIONS OF SET-VALUED MAPS(*)

by
H. Ben-El-Mechaiekh
Department of Mathematics, Brock University, St Catharines, Ontario L2S 3A1, Canada

INTRODUCTION

Our concern is to give an elementary proof of a fairly general fixed point theorem for compositions of convex valued maps defined on a general extension space containing the locally convex and non-necessarily locally convex cases, namely on an ANES(compact) (approximate neighborhood extension space for compact spaces.) For the smaller class of NES(compact) (neighborhood extension spaces), a fixed point theorem for composition of acyclic valued maps was formulated by Fournier-Gòrniewicz [2] and extended to more general classes of maps by Gòrniewicz-Granas [4]. In each case, the proof relies on a sophisticated homological machinery and, since the class NES(compact) does not contain convex subsets of locally convex spaces, the Fan-Glicksberg-Himmelberg theorem was not included. Our result contains the Fan-Glicksberg-Himmelberg theorem and recent results of M. Lassonde [7].

1. DEFINITIONS

Given a class A of set-valued maps (simply called *maps*), we define:
$A(X,Y) = \{A : X \to Y \mid A \in A\}$; $A(X) = A(X,X)$; $A_c = \{A = A_m A_{m-1} \ldots A_1 \mid A_i \in A\}$.

In what follows, X and Y are subsets of topological vector spaces E and F. Define the class:

$$A \in K(X,Y) \Leftrightarrow \text{(i) } A \text{ is usc; (ii) } A \text{ has non-empty convex compact values.}$$

Let's recall some notions about extension spaces: *(i) A space Y is a neighborhood extension space for Q if for any pair (X,K) in Q with $K \subset X$ closed and any continuous function $f_0 : K \to Y$ there is a continuous neighborhood extension $f : U \to Y$ of f_0 over a neighborhood U of K in X. The class of neighborhood extension spaces for Q will be denoted by NES(Q). (ii) A space Y is an approximate neighborhood extension space for Q if for a given covering $\alpha \in Cov(Y)$ and for any pair (X,K) in Q with $K \subset X$ closed and any continuous function $f_0 : K \to Y$ there is a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_{\alpha}|_K$ anf f_0 are α-close. The class of approximate neighborhood extension spaces for Q will be denoted by ANES(Q).*
Clearly $NES(Q) \subset ANES(Q)$. We give now some examples of extension spaces (for a more detailed list cf [5]).

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PROPOSITION (1.1). *(i) every convex subset of a locally convex t.v.s. (or of a vector space with the finite topology) is ES(metric) and therefore NES(metric) (Dugundji) ; (ii) any normed space E is ES(compact) and therefore NES(compact); every complete metric linear space admissible in the sense of Klee (cf [5]) is NES(compact) (in particular,* L^p *for* $p \leq 1$*, and the space* $\mathcal{M}$ *of measurable functions are NES(compact)); (iii) ANR = ANR(metric)* $\subset$ *NES(compact) (therefore any CW-complex is NES(compact); and if E is a locally convex t.v.s.,* $C = \bigcup_{i=1}^{n} C_i$ *with* C_i *closed in E is metrizable then C, being an ANR is an NES(compact); (iv) every open subset and every convex subset of a locally convex t.v.s. are ANES(compact).*

2. FIXED POINT FOR COMPOSITIONS OF MAPS

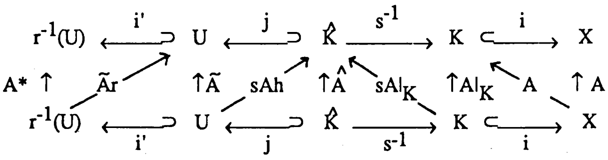
The starting is the following result, its proof is based on the existence of an approximate selection (cf [1]) and on an induction argument due to M. Lassonde [7]:

THEOREM (2.1). *Let X be a convex compact subset of a locally convex t.v.s. E and* $A \in K_c(X)$ *(the intermediate spaces being arbitrary t.v.s.), then* $Fix(A) \neq \emptyset$.

Since every Tychonoff cube $T \in \mathcal{F}_{K_c}$ and knowing that every compact space is homeomorphic to a closed subset of the Tychonoff cube, this result extends to the wider class of ES(compact) by a standard procedure (see e.g. Fournier-Gorniewicz [2], A. Granas [5]). Before proving the fixed point theorem for K_c-maps defined on an NES(compact), let's recall that the linear envelope Span(K) of a compact subset K of a t.v.s. is Lindelöf (hence normal) and therefore, if T is a Tychonoff cube, then it is a retract of Span(T). Using the fact that a compact subset of an open set in a locally convex space can be approched by a polyhedron (via the so called Schauder projection) we easily prove that open subsets of locally convex spaces have the fixed point property for the class K_c. We are now ready to prove our main results.

THEOREM (2.2). *If* $X \in$ *NES(compact) and* $A \in K_c(X)$ *is compact then* $Fix(A) \neq \emptyset$.

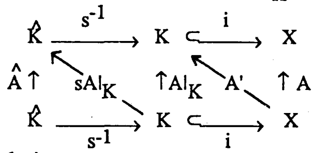
PROOF. Let $A(X) \subseteq K$, $\hat{K}$ a closed subset of the Tychonoff cube T and $s : K \leftrightarrow \hat{K}$ a homeomorphism . Since $X \in$ NES(compact), let U be an open neighborhood of $\hat{K}$ in T and $h : U \to X$ be a continuous extension of $is^{-1} : \hat{K} \to X$ on U; hence, if $\hat{K} \overset{j}{\hookrightarrow} U$ is the natural imbedding, then $hj = is^{-1}$. Consider now Span(T) in a locally convex t.v.s. containing T, then there exists a retraction $r : Span(T) \to T$. We can now build the following commutative diagram



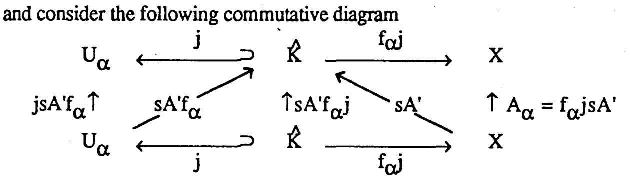
where $\hat{A}$, $\tilde{A}$ are the appropriate maps, $A^* = i'\tilde{A}r = i'jsAhr \in K_c(r^{-1}(U))$ is compact and has a fixed point. Now since $Fix(A^*) \neq \emptyset$ then $Fix(A) \neq \emptyset$.♦

THEOREM (2.3). *If $X \in ANES(compact)$ and $A \in K_c(X)$ is compact then $Fix(A) \neq \emptyset$.*[(1)]

PROOF. Let $A(X) \subseteq K$, $\hat{K}$ a closed subset of the Tychonoff cube T and $s : K \leftrightarrow \hat{K}$ a homeomorphism. We have the following commutative diagram with $A = iA'$ and $\hat{A} = sA|_K s^{-1}$



Let $\alpha \in Cov(X)$ and consider $is^{-1} : \hat{K} \to X$; since $X \in ANES(compact)$, there exist an open neighborhood U_α of $\hat{K}$ in T and $f_\alpha : U_\alpha \to X$ a continuous function such that f_α and is^{-1} are α-close on $\hat{K}$; let $\hat{K} \overset{j}{\hookrightarrow} U_\alpha$ be the natural imbedding and consider the following commutative diagram



Since $U_\alpha \in NES(compact)$, then $Fix(jsA'f_\alpha) \neq \emptyset$ and therefore $Fix(A_\alpha) \neq \emptyset$. But $f_\alpha j$ and is^{-1} are α-close i.e. A_α and A are α-close and therefore A has an α-fixed point. Since α is arbitrary, a compactness argument ends the proof.♦

This theorem contains the results of Lassonde [7], the Fan-Glicksberg-Himmelberg fixed point theorem and holds for some spaces without linear functionals (hence not locally convex).

NOTES.

(1) Theorem (2.3) was recently formulated for an abstract class of maps determined by morphisms and containing K_c by Gòrniewicz-Granas using homolgical methods [8].

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