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SOME F_σ–SETS IN L(E,F) FOR THE WEAK OPERATOR TOPOLOGY.[1]

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We shall write E,F for Banach spaces and B_E for the unit ball of E. We denote by L(E,F) the space of continuous linear mappings (operator, for short) from E to F. For a subspace S(E,F) of L(E,F) we shall denote by $S^w(E,F)$ (resp., $S^s(E,F)$) the space S(E,F) endowed the weak operator topology (resp., the strong operator topology), i.e., the topology on S(E,F) induced by the product topology of F^E when F is endowed with its weak topology (resp., its norm topology).

In this paper we prove that S(E,F) is an F_σ–set of $L^w(E,F)$ for some known subspaces S(E,F) of L(E,F); an example shows that this is not always the case.

For $0<p<\infty$, an operator S from E to F is said to be absolutely p–summing (see, for ex., (1)) if there is a constant C such that for each finite number of points $x_1,\ldots,x_n$ of E one has

$$\left[\sum_{i=1}^{n}\|Sx_i\|^p\right]^{1/p} \leq C\cdot\sup\left\{\left[\sum_{i=1}^{n}|x^*(x_i)|^p\right]^{1/p} / x^*\in B_{E^*}\right\}$$

We shal denote by $P_p(E,F)$ the set of absolutely p–summing operators from E to F.

For a finite operator $S\in L(E,E)$ which has a representation $S=\Sigma_{i=1}^n x_i^*\otimes x_i$ where $x_i^*\in E^*$ and $x_i\in E$, for $1\leq i\leq n$, the trace of S is

$$tr(S)=\Sigma_{i=1}^n x_i^*(x_i).$$

An operator $S:E\longrightarrow F$ is said to be integral if there is a constant C such that

$$|tr(SL)| \leq C\cdot\|L\|$$

for each finite operator $L:F \longrightarrow E$ (see (1)). We denote by $I(E,F)$ the set of this operators.

We shall say that an operator $S:E \longrightarrow F$ factors through a Hilbert space if there is a Hilbert space H and operators $B:E \longrightarrow H$ and $A:H \longrightarrow F$ such that $S=AB$. We denote by $H(E,F)$ the set of these operators. It is well known (2, p.25) that $S \in H(E,F)$ iff there is a constant C such that for every $n \in \mathbb{N}$ and every orthogonal matrices (a_{ij}) we have

$$\Sigma_{i=1}^{n} \| \Sigma_{j=1}^{n} a_{ij} Sx_j \|^2 \leq C^2 \cdot \Sigma_{j=1}^{n} \|x_j\|^2.$$

for all $x_1,\dots,x_n$ in E.

We write D for the product $\{-1,1\}^{\mathbb{N}}$ endowed with its Borel σ–field, $\{-1,1\}$ being considered discret, and the product probabilility P when we consider on $\{-1,1\}$ the probability Q defined by $Q(\{-1\})=Q(\{1\})=1/2$. We write $(r_n)_n$ for the (Bernouilli) sequence of the projection from D onto its factors.

For $1<p\leq 2$ we say that an operator $S:E \longrightarrow F$ is of type p if there is a constant C such that for any $n \in \mathbb{N}$ and any $x_1,\dots,x_n \in E$ we have

$$\left[\int_D \| \Sigma_{i=1}^{n} r_i(t) \cdot Sx_i \|^2 \, dP(t)\right]^{1/2} \leq C \cdot \left[\Sigma_{i=1}^{n} \|x_i\|^p\right]^{1/p}.$$

We shall denote by $T_p(E,F)$ the space of all operators of type p from E to F. The dual Banach space of $L_2(P_n;F)$ is $L_2(P_n;F^*)$, where P_n $(n \in \mathbb{N})$ denotes the uniform probability on $\{-1,1\}^n$.

For $2\leq q<\infty$ we shall say that an operator $S \in L(E,F)$ is of cotype q if there is a constant C such that for any $n \in \mathbb{N}$ and any $x_1,\dots,x_n \in E$ we have

$$\left[\sum_{i=1}^{n} \|Sx_i\|^q\right]^{1/q} \leq C \cdot \left[\| \sum_{i=1}^{n} r_i x_i \|_{L_2(E)}\right].$$

We shall write $C_q(E,F)$ for the space of these operators. The first member in the precedent inequality is the norm of $(Sx_1,\dots,Sx_n)$ in the Banach space $l_q^n(F)$ whose dual is $l_{q^*}^n(F^*)$, q^* being the conjugate exponent to q.

We refer to (2) for the definitions and some results related with type and cotype and factorization through a Hilbert space.

We have then the following result.

Theorem: For $S=P_p$, I, H, T_p, C_q, S(E,F) is an F_σ–set in L(E,F) for the weak operator topology.

The following example shows that we cannot expect similar results for the space of weakly compact operators.

EXAMPLE: The space of weakly compact operator $W(l_1,F)$ from l_1 to F is an F_σ–set of $L^w(l_1,F)$ if and only if F is reflexif. A sequence in the unit ball of F with no weakly convergent subsequences and a Baire category argument gives the proof.

REMARKS: 1) The proof in this example shows even that $W(l_1,F)$ for F non–reflexif is not an F_σ–set in $L(l_1,F)$ for the strong operator topology.

2) The method in the example shows even that the space of compact operators from l_1 to F is an F_σ–set for the weak operator topology iff dim F $<\infty$.

3) As a consequence of the results contained in this paper we have, for example, that $S^w(E,F)$ (resp., $S^s(E,F)$) is $\mathcal{K}$–analytic whenever $L^w(E,F)$ (resp., $L^s(E,F)$) is $\mathcal{K}$–analytic if $S=P_p$,I,H,T_p or C_q. The reader can find in (3) some results about the $\mathcal{K}$–analyticity of L(E,F) for the weak and strong operator topology.

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