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δ-DIMENSION: A TRANSFINITE EXTENSION OF THE SMALL INDUCTIVE DIMENSION

by

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We define the δ-dimension as a transfinite extension of the small inductive dimension using the same method in [7] and [3] for defining the D-dimension and the d-dimension respectively. We state that for every normal space X we have $d(X)\leq\delta(X)\leq D(X)$ and that for a perfectly normal , strongly paracompact space X, if trInd(X) exists, $\mathrm{trind}(X)\leq\mathrm{trInd}(X)\leq\delta(X)$. For a normal space X which is the union of a countable family of closed strongly pseudometrizable subspaces, we have $\delta(X)=d(X)=D(X)$. Also, we give a relation of this dimension with a problem of Y.Hattori (see [6]).

Definition 1. *Let X be a regular space; if β is an ordinal number or $\beta=-1$, a* ***β-δ-representation*** *of X is every expression:*

$$X = \bigcup_{0\leq\alpha\leq\gamma} A_\alpha$$

where:

a) For $0\leq\alpha\leq\gamma$, A_α is a closed set of X such that $\mathrm{ind}(A_\alpha)<\infty$.

b) $\gamma=\lambda(\beta)$ and $\mathrm{ind}(A_\gamma)=n(\beta)$.

c) For $\mu\leq\gamma$ the set $\bigcup_{\mu\leq\alpha\leq\gamma} A_\alpha$ is a closed set of X.

d) For each $x\in X$ there is a greatest ordinal μ such that $x\in A_\mu$.

Definition 2. *For a regular space X, we define the* ***δ-dimension*** *of X, $\delta(X)$, as an ordinal number, -1 or Δ verifying:*

δ.1. $\delta(X) = -1$ if and only if $X = \emptyset$.

δ.2. For $X \neq \emptyset$, $\delta(X)$ is the smallest ordinal β (if it exists) such that X has a β-δ-representation.

δ.3. If $X\neq\emptyset$ and for every ordinal number β there is no β-δ-representation of X, then $\delta(X) = \Delta$.

The δ-dimension of a regular space X is a transfinite extension of the small inductive dimension because if both are finite, they coincide. However, the transfinite extensions trind and δ of the dimension ind differs in the general case:

Example. The space S_{ω_0+3} defined by Smirnov in [10] is a compact metric space such that $\mathrm{trind}(S_{\omega_0+3}) < \omega_0+3$ (see [5]) while $\mathrm{trInd}(S_{\omega_0+3}) = \omega_0+3$. From theorem 3 in [3] we have:

$$\mathrm{trind}(S_{\omega_0+3}) < \mathrm{trInd}(S_{\omega_0+3}) \leqslant D(S_{\omega_0+3})$$

Since S_{ω_0+3} is a compact metric space, $\delta(S_{\omega_0+3}) = D(S_{\omega_0+3})$; therefore, $\mathrm{trind}(S_{\omega_0+3}) \neq \delta(S_{\omega_0+3})$

As an immediate consequence of the coincidence of the dimension functions ind(X), Ind(X) and dim(X) in the class of separable metric spaces, the transfinite dimensions $\delta(X)$, d(X) and D(X) coincide on that class of spaces. In the next examples we see that outside this class of spaces the transfinite dimensions $\delta(X)$, d(X) and D(X) can be different:

Examples. 1. The space X defined in [4] (2.2.1) verifies $\mathrm{ind}(X)=\delta(X)=0$ and $\mathrm{Ind}(X)=D(X)=1$.

2. The Roy's space Δ ([9]) is a completely metrizable space such that $\mathrm{ind}(\Delta)=\delta(\Delta)=0$ and $\dim(X)=d(X)=1$.

3. In [8] we have a completely normal space X such that $\mathrm{Ind}(X)=\dim(X)=0$ and for each $n=0,1,2,\ldots$ there is a subspace X_n of X such that $\dim(X_n)=\mathrm{Ind}(X_n) = n$. Since for every normal space X we have $\mathrm{ind}(X)\leqslant \mathrm{Ind}(X)$, we have $\mathrm{ind}(X_n)\leqslant 0$ for $n=0,1,2,\ldots$

Let $Z = \bigoplus_{n=0}^{\infty} X_n$; now, $\dim(Z)=\mathrm{Ind}(Z)=\infty$; i.e., Z is a space in the class S (see [5]) and, therefore, $D(Z)=d(Z)=\omega_0$ (see [2] and [3]) while, from [4] (theorem 1.3.1 and remark 1.3.2), we have $\mathrm{ind}(Z)=\delta(Z)=0$.

Proposition 1. *For every normal space X, $\delta(X)\leqslant D(X)$.*

Proposition 2. *Let X be a strongly paracompact, strongly hereditarily normal space; then, $\delta(X)=D(X)$.*

Proposition 3. *For a normal, strongly paracompact space X we have* $d(X)\leqslant\delta(X)$.

Proposition 4. *If X is a normal space such that it is the union of a countable family of closed strongly pseudometrizable subspaces we have* $\delta(X)=d(X)=D(X)$.

Proposition 5. *Let X be a perfectly normal, strongly paracompact space such that trInd(X) exists. Then,* $trind(X)\leqslant trInd(X)\leqslant\delta(X)$.

Related with a question of Y. Hattori in [6] about the transfinite dimension w-Ind defined by P. Borst in [1] we have:

Proposition 6. *Let X be a compact metric space. Then,* $w\text{-}Ind(X)\leqslant\delta(X)$

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