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THE LIMIT SET OF FUCHSIAN AND KLEINIAN GROUPS*

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The theory of Fuchsian and Kleinian groups was created and developed by Poincaré and Klein in the later part of last century. It was quite dormant until the sixties when Ahlfors and Bers brought it back into the main stream of mathematical research.

We shall be concerned chiefly with one specific aspect of this topic. Nonetheless even surveying this confined area would be quite impossible and what I have in mind is more like a leisurely trip through some of my favorite spots.

Here is a guide:

1. Definitions.
2. Examples.
3. The limit set. Hopf-Tsuji-Sullivan theorem.
4. The exponent of convergence.
5. Spectral theory. An illustration.
6. Diophantine analysis.
7. Action on S^1. Mostow's rigidity theorem.

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1. DEFINITIONS.

We shall denote by Δ the unit disk in the complex plane $\mathbf{C}$ and by U its upper half plane. By $\mathbf{M}(\Delta)$ and $\mathbf{M}(U)$ we shall mean respectively the group of all sense preserving Mobius transformations from Δ onto itself and from U onto itself. Notice that $\mathbf{M}(U) = SL(2,\mathbf{R})/\{\pm I\} = PSL(2,\mathbf{R})$.

A **Fuchsian group** is a subgroup of $\mathbf{M}(\Delta)$ (or $\mathbf{M}(U)$) which acts discontinuously on Δ (or U). This is equivalent to being a discrete subgroup of $SL(2,\mathbf{R})/\{\pm I\}$.

A fundamental fact, which was discovered by Poincaré, is that $\mathbf{M}(\Delta)$ is **also** the full group of direct isometries of Δ endowed with the usual model of hyperbolic geometry i.e. the Riemannian metric

$$ds = \frac{2}{1-|z|^2}|dz|$$

(the factor 2 is there so that it has curvature -1).

By a Kleinian group it was usually meant a subgroup of $SL(2,\mathbf{C})/\{\pm I\} = \{$ Mobius transformation of the Riemann sphere S^2) which acts discontinuosusly somewhere on S^2. Thus, Fuchsian groups are just a particular case of Kleinian groups. Poincaré recognized though that Kleinian groups should be considered as higher-dimensional generalizations of Fuchsian groups. This is the case since Mobius transformations are just products of reflections on circles C of the sphere S^2 and any such reflection can be extended in a canonical way to act on the unit ball of $\mathbf{R}^3$ (simply consider the reflection on the sphere which intersects S^2 orthogonally on C).

From now on the term Kleinian group will mean a subgroup of Mob(B^n) which acts discontinuonsly in $B^n, n \geq 3$. (It may act discontinuonsly nowhere on S^{n-1} and so for n=3 this is a more general concept that the one mentioned above.)

2. EXAMPLES.

Let R be the rectangle $\{z = x+iy;\quad 0 < x < a,\quad 0 < y < b\}$. By the Riemann mapping theorem there is a conformal mapping from R onto U so that the vertices are carried onto the points $0, 1, \infty$ and, say, k. If you are Schwarz you would use a certain principle to succesively reflect f and obtain in this way a meromorphic map F. Moreover, if Γ is the group generated by the translation by **2a** and **2ib** then the map F is G-equivariant i.e. F is an elliptic function. As a matter of fact F is a Weierstrass $\mathcal{P}$-function.

Suppose now that T is a "triangle" whose sides are segments of straight lines or circles. As above we have a conformal mapping f from T onto U so that the vertices are carried onto the points 0,1 and ∞. Let G be the group generated by reflections on the sides of T and let Γ be the subgroup of words of even length (the elements of Γ are sense preserving). If we try to apply the process above to get a meromorphic Γ-equivariant function F we readily see that the angles at the vertices of T must be of the form $\pi/p, \pi/q, \pi/r$ where p,q,r are positive integers (or ∞). The triangle T would be spherical, euclidean or hyperbolic according to $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1, = 1$ or < 1. For the first option to occur one must have for (p,q,r) the value (2,2,n), $n \geq 1$; (2,3,3); (2,3,4) or (2,3,5) which correspond to G being a dihedral group, or the group of symmetries of the tetrahedron, octahedron/ cube, icosahedron/ dodecahedron. All are finite groups. For the second option only the triples (3,3,3); (2,4,4); (2,3,6); (2,2,∞) may occur; the first three give the usual triangle tesselations of the plane and the corresponding F are **a** $\mathcal{P}'$, **a** $\mathcal{P}^2$, and **a** $\mathcal{P}'^2$ while the last one has a half strip as triangle and cos z as F.

But the third option gives us a whole zoo: the triangle groups and functions of Schwarz. Now G(T) is a tesselation of Δ by congruent hyperbolic triangles, and Γ is a our first example of Fuchsian group. A fundamental region for its action is obtained by adjoining two adjacent triangles of G(T).

The hyperbolic area of T may be calculated by the Gauss-Bonnet theorem and it is

$$\pi(1 - 1/p - 1/q - 1/r).$$

The smallest possible value is $2.\frac{\pi}{21}$ which occurs when (p,q,r)= (2,3,7). This is the familiar picture of the densest triangle tesselation of the hyperbolic plane.

Notice that the equivariant meromorphic function F has branching of order (p,q,r) at the vertices of the tesselation but maps the fundamental region in a 1-1 fashion onto a slit plane. In the limiting case, i.e. when p=q=r=∞, the triangle T has vertices at the hyperbolic ∞ (i.e on the circunference of Δ); this means that in this case there is no branching (it has been pushed away) and F is now the familiar universal cover of the three-punctured sphere. An immediate corollary is Picard's theorem.

Actually, by the uniformization theorem there are only three simply connected Riemann surfaces: $\Delta, \mathbf{C}$ and S^2. Every Riemann surfaces can be represented as a quotient of a simply conected one, X, by a fixed point free discontinuous group of conformal automorphism of X. With $X = S^2$ or $X = \mathbf{C}$ this produces only $S^2, \mathbf{C}, \mathbf{C} \setminus \{0\}$ and the tori, therefore, any other Riemann surface S is represented as

$$S \simeq \Delta/\Gamma$$

where Γ is a Fuchsian group (without elliptic elements).

Since the elements of Γ are also hyperbolic isometries this endows S with a complete metric of constant negative curvature. More generally, if $\mathcal{M}$ is a complete Riemannian n-manifold of constant negative sectional curvature then the universal cover is also of constant negative curvature and complete and so it is hyperbolic n-space. Thus

$$\mathcal{M} = B^n/\Gamma$$

where Γ is a Kleinian group.

We can add branching data to the uniformization process. Let us consider one very particular instance of this. Let us mark in the sphere S^2 n points $a_1, ..., a_n, n \geq 3$, and assign positive integers $p_1 ... p_n$ to them with the requirement that $\sum_{i=1}^{n} \frac{1}{p_i} < 1$. Then there is a Fuchsian group Γ(now with elliptic elements) so that

$$\Delta/\Gamma \simeq S^2$$

and the projection $\Pi : \Delta \to S^2$ is branched of order p_i over the preimages of $a_i, i = 1, ..., n$. The inverse Z of π is defined locally on $S^2 \setminus \{a_1 ... a_n\}$. If we let $\varphi = \frac{\{\pi,z\}}{\pi'^2}$, where $\{\pi, z\}$ is the Schwarzian derivative of π, then it can be seen that Z is the quotient of a fundamental pair of solutions of the differential equation.

$$w'' + \varphi.w = 0 \tag{1}$$

Under special circumstances this whole process can be reversed, and starting from an equation like (1) obtain the group and the covering. The hypergeometric equation lead Schwarz to the triangle groups and Fuchs papers on linear differential equations were an inspiration for Poincaré work on Fuchsian groups. It was him who coined the term.

The most important example of Fuchsian group is the modular group SL(2,**Z**). It is generated by $z \to z + 1$ and $z \to -1/z$. Its fundamental region R is the familiar modular picture.

$$R = \{z : \quad |\text{Re} \quad z| < 1/2, \quad \text{Im} \quad z > 0, \quad |z| > 1\}.$$

This is equivalent to the fact that any positive definite quadratic form

$$ax^2 + 2bxy + cy^2; \qquad x, y \in \mathbf{R}$$

is unimodulary equivalent to another

$$a'x^2 + 2b'xy + c'y^2$$

where

$$2|b'| \leq a' \leq c'.$$

The corresponding analogue for hermitian posivive definite quadratic forms

$$"a|x|^2 + bx\bar{\tau} + \bar{b}\bar{x}\tau + c|\tau|^2$$

$$is \quad equivalent \quad to$$

$$a'|x|^2 + b'x\bar{\tau} + \bar{b}'\bar{x}\tau + c'|\tau|^2$$

where

$$2 \quad \max\{|\mathrm{Re} \quad b|, |\mathrm{Im} \quad b|\} \leq a' \leq c'",$$

is provided by the Picard group $SL(2, \mathbf{Z} + i\mathbf{Z})$ acting on $\mathbf{R}^2 \times \mathbf{R}^+$ by means of Poincaré's device.

Our final example is the Schottky groups. Consider a configuration of four (in general $2g, g > 1$) mutually disjoint disks D_i in the plane. Let T (respectively S) be the Mobius transformation which takes the outside of D_1 (resp., D_3) onto the inside of D_2 (resp. D_4). The group G generated by S and T is a Kleinian group. A fundamental region for its action **on** $\mathbf{S}^2$ is the region autside the four disks. Notice that S^2/G is a compact Riemann surface of genus 2.

3. THE LIMIT SET. THE HOPF-TSUJI-SULLIVAN THEOREM.

Let Γ be a Kleinian group in $B^n, n \geq 2$. The orbit of $0, \Gamma(0)$, accumulates only on $\partial B^n = S^{n-1}$. The derived set of $\Gamma(0)$ is called the (topological) **limit set** $\Lambda(\Gamma)$ of Γ. A bit of hyperbolic geometry shows that if any other orbit is used then the same set $\Lambda(\Gamma)$ is obtained.

The limit set has either 0, 1 or 2 points or else it is a perfect set. Those groups with finite limit sets are called elementary and they are readily classified. If Γ is not elementary then $\Lambda(\Gamma)$ is a minimal Γ-invariant closed set in $\bar{B}^n$ (or S^{n-1}).

$\Lambda(\Gamma)$ may be the whole of S^{n-1} but if this is not so then $\Lambda(\Gamma)$ is a totally disconnected nowhere dense Cantor like set, or better, to say it at least once, a **fractal** .

We shall be mostly concerned with studying the **size of** $\Lambda(\Gamma)$ and also the dynamics of **the action** of Γ **on** $\Lambda(\Gamma)$.

The main developers of this theory are A. Beardon, S. Patterson, and D. Sullivan.

Example: Consider the group generated by $z \to \frac{z+\lambda}{1+z\lambda}$ and $z \to \frac{z+i\lambda}{1-z(i\lambda)}$. Call it $G(\lambda)$. If $1 > \lambda \geq \frac{\sqrt{2}}{2}$ then $G(\lambda)$ is a Fuchsian group acting on Δ. When $\lambda = \frac{\sqrt{2}}{2}$ then $G\left(\frac{\sqrt{2}}{2}\right)$ has the whole circle as limit set, if $\lambda > \frac{\sqrt{2}}{2}$ then $G(\lambda)$ has fractional dimension one can show that

$$\frac{K}{\log \dfrac{1+\lambda}{1-\lambda}} \geq D\left(\Lambda(G(\lambda))\right) \geq \frac{\log 3}{\log \dfrac{1+\lambda}{1-\lambda}}$$

where K is an absolute constant, and that

$$\lim_{\lambda \to 1} D\left(\Lambda(\lambda)\right) \log \frac{1+\lambda}{1-\lambda} = \log 3.$$

A group Γ is geometrically finite if it admits a fundamental polyhedron with finitely many faces. In the case $n = 2$ a theorem due to Siegel asserts that Γ is geometrically finite iff Γ is finitely generated. But for $n \geq 3$ not every finitely generated group is geometrically finite (Greenberg) though the converse holds.

Q1. Ahlfors' area conjecture (1964). For a finitely generated Kleinian group Γ in B^n.

$$\mathcal{H}^{n-1}(\Lambda(\Gamma)) = 0 \quad \text{or} \quad \Lambda(\Gamma) = \mathcal{S}^{n-1}.$$

This is probably the most fundamental problem in our topic of interest. We shall presently show that the conjecture holds for geometrically finite groups and it is known that for certain limits of geometrically finite groups the conjecture holds (Thurston).

Let us introduce now an important subset of $\Lambda(\Gamma)$. It is the so-called **conical limit set** $C(\Gamma)$. It is defined as follows:
$\xi \in C(\Gamma)$ if a subsequence of $\Gamma(0)$ lies on some **Stolz** cone at ξ.

This is independent of the orbit used. Notice that $C(\Gamma)$ is Γ-invariant.

Example: $\Gamma = SL(2, \mathbf{Z})$ then $\Lambda(\Gamma) = \mathbf{R}$, $C(\Gamma) =$ irrationals. We shall discuss this example later, but let us remark now that the situation it describes is quite general: (Beardon-Maskit) **If Γ is geometrically finite then $\Lambda(\Gamma) \setminus C(\Gamma)$ is countable.**

With no finiteness assumption $C(\Gamma)$ always satisfies a fundamental dicothomy:

$$(*) \qquad \text{either} \quad \mathcal{H}^{n-1}(C(\Gamma)) = 0 \quad \text{or} \quad \mathcal{H}^{n-1}(\mathcal{S}^{n-1} \setminus C(\Gamma)) = 0$$

We shall see later how this relates to several other dicothomies, but observe that an inmediate consequence is that Ahlfors conjecture holds for geometrically finite groups, (use Beardon-Maskit), and that it holds for $n = 2$ in full generality (use Siegel's result). The proof of (*) is so simple that we can not resist giving it.

Proof of (*). Start with $n = 2$.

Let u be the Poisson extension of $\chi_{C(\Gamma)}$. Then since $C(\Gamma)$ is Γ-invariant we have that u is Γ-equivariant, i.e. $u_0\gamma = u$, $\forall \gamma \in \Gamma$. Now for a.e. $\xi \in C(\Gamma)$ we have $\lim_{z > \xi} u(z) = 1$, where $z > \xi$ means z approaches ξ within a Stolz angle, but then we deduce, if $|C(\Gamma)| > 0$, that $u(0) = 1$ and so $|S^1 \setminus C(\Gamma)| = 0$. For $n = 3$ the proof is the same but one should use the hyperbolic Poisson Kernel.

Consider now the (absolute Poincaré) series.

$$S = \sum_{g \in \Gamma} (1 - |g(0)|)^{n-1}$$

If this series converges/diverges then the group is called of convergence/divergence type. This is a relevant distinction. For one thing if $S < \infty$ then necessarily $\mathcal{H}^{n-1}(C(\Gamma)) = 0$. This is the easy half of the Borel-Cantelli Lemma. It turns out that the other half works here too. This is part of the following beautiful theorem:

Hopf-Tsuji-Sullivan

Let Γ be fixed point free Klenian group on B^n, and let $M = B^n \setminus \Gamma$ be endowed with the proyection of the hyperbolic metric. Then the following statements are equivalent:

(i) Γ **is of divergence type**

(ii) $\mathcal{H}^{n-1}\left(S^{n-1}, C(\Gamma)\right) = 0$

(iii) M **does not possetss a Green's function for the Laplace-Beltrami operator**

(iii') **There are no positive non-constant superharmonic functions on M.**

(iii") **Brownian motion on M is recurrent**

(iv) **The geodesic flow on the unit sphere bundle is ergodic with respect to Liouville's measure**

(v) **The diagonal action of Γ on $S^{n-1} \times S^{n-1}$ is ergodic.**

We should remark that Γ never acts ergodically on $(S^{n-1})^k$, if $k \geq 3$.

While the equivalence of (iii') and (v) is a deep result the parallel fact that action on S^{n-1} is ergodic iff there are no bounded (or positive) non-constant harmonic functions is trivial. In the case $n = 2$ only after a long search it was found that there are Riemann surface with Green's function but such that all bounded harmonic functions are constants. For plane domains these two properties are equivalent.

Q2. Is there a group characterization like (i) or (ii) of the Liouville property: there are no non-constant bounded harmonic functions?.

Varopoulos has given a partial solution to this question, namely:

if Γ is a normal sobgroup of a finitely generated Fuchsian group then $\Delta \setminus \Gamma$ has Liouville's property iff $|V(\Gamma)| = 2\pi$, where $V(\Gamma)$ is defined as those $\xi \in S'$ so that for same sequence of $\gamma_n \in \Gamma$ h-dist $(\gamma_n(0), \quad$ ray at $\quad \xi) = O\left(\frac{1}{h-dist(\gamma_n(0),0)}\right)$

If we further assume that $n = 2$ and that M is a plane domain then we add to these conditions the theorem of R. Nevanlinna stating that the following are equivalent

(iii) M does not possess a Green's function

(vi) ∂M has zero logarithmic capacity

(vii) The universal cover (now a function) π does not belong to the Nevanlinna class i.e.,

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ \left|\Pi(\mathrm{re}^{\theta}\right| d\theta = \infty.$$

Example: Consider a totally disconnected compact set $E \subset \mathbf{C}$ of positive logarithmic capacity let $M = \mathbf{C}^1 \setminus E$ and let Γ be the covering group of M. Thus $\Delta/\Gamma = M$. Then Γ is of convergence type and so $C(\Gamma)$ has zero length while $\Lambda(\Gamma) = S^1$.

We may have stronger examples where $C(\Gamma)$ **has fractional dimension but** $\Lambda(\Gamma) = S^1$, one simply takes $E \subset [0,1]$ to be **the** Cantor set, or any fractal of positive dimension. This depends on the work in 5.

4. EXPONENT OF CONVERGENCE.

The condition of convergence or divergence says, as we have seen, quite a bit about the size of the quotient manifold (or the boundary of a Riemann surface). A simple packing argument in hyperbolic space shows that for every group

$$\sum_{\gamma\in\Gamma}(1-|\gamma(0)|)^{\alpha} < \infty$$

for every $\alpha > n-1$.

It is natural to introduce, as was done by Beardon, the so-called **exponent of convergence** as

$$\delta(\Gamma) = inf\left\{t : \sum_{\gamma\in\Gamma}(1-|\gamma(0)|)^{t} < \infty\right\}$$

. Thus $\delta(\Gamma) \leq n-1$, and, but for a few exceptions $\delta(\Gamma) > 0$.

By the very definitions one sees that

$$\delta(\Gamma) \geq Dim(C(\Gamma)).$$

For $n = 2$ we have the following result (Patterson-Sullivan)

$$(*) \qquad \delta(\Gamma) = Dim(C(\Gamma))$$

and in particular

$$\delta(\Gamma) \leq Dim(\Lambda(\Gamma)).$$

and so for geometrically finite Fuchsian groups

$$\delta(\Gamma) = Dim(\Lambda(\Gamma)).$$

As far as I know the following question remains open.

Q3. Does (*) hold for Kleinian groups, $n \geq 3$?.

A consequence of (*) which we have already mentioned in that for finitely generated Fuchsian groups either $\Lambda(\Gamma) = S^1$ or else $Dim(\Lambda(\Gamma)) < 1$. This is a theorem of Beardon. It also holds for geometrically finite Kleinian groups and $n = 3$. This is due to Sullivan and Tukia.

Q4. Does Beardon-Sullivan result hold for arbitrary n?.

In this direction Phillips-Sarnak and Doyle have shown that if Γ is any Schottky group, $n \geq 3$, then

$$Dim(\Lambda(\Gamma)) \leq C_n < n-1$$

C_n depends only on n. This result is plainly false when $n = 2$, as the group $G\left(\frac{1}{\sqrt{2}}\right)$ shows.

The proof of the Patterson-Sullivan theorem above depends on the notion of conformal density and the existence of the Patterson measure.

A conformal density of dimension α is a probability measure μ on $\Lambda(\Gamma)$ which satisfies

$$g_*(\mu) = |g'|^\alpha \mu; \forall g \in \Gamma$$

where $|g'|$ is the linear distortion of g.

The result is that

(i) There is a conformal density of dimension $\delta(\Gamma)$.
(ii) Any conformal density has dimension $\geq \delta(\Gamma)$.

A conformal density of dimension $\delta(\Gamma)$ is a **Patterson measure**. The invariance property tells a lot about the fractal nature of $\Lambda(\Gamma)$. Here is an outline of how it is obtained. Assume for simplicity that $\sum (1 - |\gamma(0)|)^\delta = \infty$.

Now let $s > \delta$ and define

$$\mu_s = \sum e^{-sd(0,\gamma(0))} \delta_\gamma(0),$$

where d means hyperbolic distance; set $\gamma_s = \frac{\mu_s}{\mu_s(\Delta)}$.

If γ is any weak limit of the γ_s, then γ is our measure. First, because $\mu_s(\Delta) \to \infty$, one sees that γ is supported on $\Lambda(\Gamma)$.

Now, if $x \in B^n$, $y \in B^n$ and y converges to $z \in S^{n-1}$ euclideanly then a calculation shows that

$$d(0,y) - d(x,y) \to \frac{1 - |x|^2}{|x - z|^2}$$

But then

$$g_*(\gamma_s) = \frac{1}{\mu_s(\Delta)} \sum_\gamma e^{-sd(0,\gamma(0))} \delta_{g^{-1}\gamma(0)} =$$
$$= \frac{1}{\mu_s(\Delta)} \sum_\gamma e^{-sd(0,g\gamma(0))} \delta_{\gamma(0)} =$$
$$= \frac{1}{\mu_s(\Delta)} \sum_\Gamma e^{-s[d(g^{-1}(0),\gamma(0))-d(0,\gamma(0)]} e^{-sd(0,\gamma(0))} \delta_{\gamma(0)}$$

Therefore $lim_{s\to\delta} \left(\frac{1-|g^{-1}(0)|}{|\cdot - g^{-1}(0)|^2}\right)^\delta \gamma_s = |g'(\cdot)|^\delta \gamma_s$.

5. SPECTRAL THEORY. AN ILLUSTRATION.

If you write

$$F(w) = \int_\Pi P(w,\rho)^{\delta(\Gamma)/n-1} d\mu(\rho),$$

where μ is the Patterson measure, and P is the hyperbolic Poisson Kernel: $P(w,\rho) = \left(\frac{1-|w|^2}{|w-\rho|^2}\right)^{n-2}$, then F is Γ-equivariant and the (hyperbolic) Laplacian of F is

$$\Delta F = \delta(\Gamma)\,(n - \delta(\Gamma))\,F$$

Also $F \geq 0$. There is a remarkable result, again due to Patterson and Sullivan, concerning the spectral theory of the Laplacian.

Let Γ be a Kleinian group and let

$$b(\Gamma) = inf\left\{\delta(\Gamma)\,(n - \delta(\Gamma))\,, \frac{n^2}{4}\right\}$$

Then:

(1) $b(\Gamma)$ is the bottom of the L^2-spectrum of $B^n/\Gamma = X$ i.e.

$$b(\Gamma) = \inf_{\rho\in C_0^\infty(X)} \frac{\int |\nabla\rho|^2 dw_X}{\int |\rho|^2 dw_X},$$

where dw_X is the volume form of the hyperbolic metric of X.

(2) $b(\Gamma)$ is the upper bound of the positive spectrum. i.e.

if $t \leq b(\Gamma)$ there exists $\rho \in C^\infty$, $\rho \geq 0; \Delta\rho = t\rho$.
if $t > b(\Gamma)$ no such ρ exists.

Let us illustrate the use of these ideas by dealing with the so-called uniformily fat sets.

We have mentioned that if a plane domain Ω has boundary of positive logarithmic capacity then the covering group Γ is of convergence type (and conversely).

For such a domain "nearly" every point on $\partial\Omega$ is regular for the Dirichlet problem. Domains for which Wiener's criteria holds in a uniformly way are interesting in function theory. Let us call a domain Ω , $\infty \in \Omega$, uniformly fat (at the boundary) if

$$(r \geq) cap(\Delta(a,r) \cap \partial\Omega) \geq cr$$

for some constant $c > 0$ and for all $r \leq diam\partial\Omega$.

It is then easy to see that if Ω is uniformly fat then some sort of uniform version of convergence type holds. Namely, define the absolute Poincaré theta/zeta function.

$$U(z) = \sum_{[\gamma]\in\Pi_1(\Omega,z)} exp(-length([\gamma]_z)),$$

where by $length([\gamma]_z)$ we mean the infimum of the lengths of the curves in $[\gamma]_z$ (= curves homotopic to γ with z fixed).

Now, if Ω is uniformly fat then $U \in L^\infty(\Omega)$. (The condition $U \not\equiv \infty$ is simply convergence type).

We have:

(*) if Ω is uniformily fat then $\delta(\Gamma) < 1$,

The converse is not true. One example is $\Omega = \Delta \setminus \{0\}$.

Proof. of (*):

The Poincaré metric of Ω is conformal with the euclidean metric:

$$ds^2 = \lambda_\Omega(z)^2 |dz|^2$$

Then:

(1) $$\Delta(log\lambda_\Omega) = {\lambda_\Omega}^2$$

This is simply a translation of the fact that ds has constant curvature -1.

It also satisfies

(2) $$\lambda_\Omega \leq \frac{2}{dist(\cdot,\partial\Omega)} \qquad (Schwarz' Lemma)$$

$$|\nabla log\lambda_\Omega| \leq \frac{100}{dist(\cdot, \partial\Omega)} \qquad (BMO). \tag{2}$$

These facts, (1) and (2), are always true. Somehow one thinks of λ_Ω as behaving like $dist(\cdot, \partial\Omega)^{-1}$, but actually this is the case exactly when Ω is uniformily fat, (Beardon-Pommerenke).

Thus, in our case:

$$\lambda_\Omega \geq \frac{c_1}{dist(\cdot, \partial\Omega)}, \tag{3}$$

for some constant c_1. Combining (2) and (3) we have

$$|\nabla log\lambda_\Omega| \leq c_2 \cdot \lambda_\Omega. \tag{4}$$

If A is a smoothly bounded domain, $A \subset\subset \Omega$, then by Green's theorem and (1) and (4) we have

$$\begin{aligned} \lambda_\Omega - area \quad of A &= \iint_A {\lambda_\Omega}^2 dxdy = \iint_A \Delta(log\lambda_\Omega) dxdy = \\ &= \int_{\partial A} \nabla log\lambda_\Omega \cdot \vec{n} |dz| \leq c_2 \int_{\partial A} \lambda_\Omega |dz| = \\ &= c_2 \cdot \lambda_\Omega - length \quad of \quad \partial A. \end{aligned}$$

Thus we have an isoperimetric inequality.

But a well known general estimate due to Cheeger tells us that

$$b \geq \frac{1}{4} \cdot \frac{1}{c - 2^2},$$

in particular, $b > 0$, and therefore $\delta < 1$.

The argument above uses quite a lot that Ω is a plane domain. Thus:

Q.5.Does it follow for a Riemann surface M that $\delta < 1$ if $U \in L^\infty$?. What about higher dimensions?.

6. DIOPHANTINE ANALYSIS.

Considere the groups $\Gamma = SL(2, \mathbf{Z})$.

If $\xi \in C(\Gamma)$ then $\exists g_n \in \Gamma$ so that $|\xi - g_n(i)| < C_\xi \quad Img_n(i)$. This implies that

$$|\xi - g_n(0)| < C_\xi Img_n(i)$$

or

$$|\xi - g_n(\infty)| < C_\xi Img_n(i)$$

If $g \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $Img(i) = \frac{1}{c^2+d^2}$.

We obtain that there exists $\alpha_n, \beta_n \in \mathbf{Z}, \quad g.c.d.(\alpha_n, \beta_n) = 1$ so that

$$|\xi - \frac{\alpha_n}{\beta_n}| < C_\xi \frac{1}{\beta_n^2}.$$

Which is Dirichlet's theorem. And conversely. Thus $C(\Gamma) = \mathbf{R} \setminus \mathbf{Q}$. It is natural to pursue this parallelism further.

Lets us recall some classical results of the metrical theory of Diophantine approximation.

(1) (Kintchine). Let w_n be a sequence of positive numbers decreasing to 0 , and such that $w_{2k} \geq Cw_k; \quad k \in \mathbf{N}$.

Define K_w as the set of points for which the inequality

$$|\xi - \frac{\alpha}{\beta}| \leq \frac{w_\beta}{\beta_2}$$

Ocurrs for infinitely many $\alpha, \beta \in \mathbf{Z}, (\alpha, \beta) = 1$. Then $|K_w| = 0$ or $|\mathbf{R} \setminus K_w| = 0$ according to $\sum w_n < \infty$ or $\sum w_n = \infty$.

(2) (Jarnik). Let J be the set of numbers ξ such that

$$|\xi - \frac{\alpha}{\beta}| > \frac{C_\xi}{\beta^2}$$

For **every** pair $(\alpha, \beta) \in \mathbf{Z}^2, \quad (\alpha, \beta) = 1$. Them

$$Dim(J) = 1$$

(3) (Jarnik-Besicovitch). Let $B_\tau, \quad \tau \geq 1$, be the set of those numbers ξ so that:

$$|\xi - \frac{\alpha}{\beta}| \leq \frac{C_\xi}{(\beta^2)^\tau}$$

occurs for infinitely many pairs $(\alpha,\beta)\in \mathbf{Z}^2$, $(\alpha,\beta)=1$. Then

$$Dim(B_\tau)=\frac{1}{\tau}$$

If $g\in SL(2,\mathbf{R})$ we associate

$$\mu(g)=Im\quad g(i)\left(=\frac{1}{c^2+d^2}\right)$$

The basic type of approximation we will be considering is the following

$$|\xi-g(i)|<C_\xi\mu(g).$$

We seek analogues of the previous theorems. This is an unfinished and, I think, attractive job.

To start off let us outline how Kintchine's result can be obtained. Let

$$A_k=\left\{\xi\in[0,1]:|\xi-\tfrac{\alpha}{\beta}|<\tfrac{w_\beta}{\beta^2},\quad for\quad some\quad \alpha,\beta\in\mathbf{Z},\beta\in(2^k,2^{k+1}),(\alpha,\beta)=1\right\}.$$

The sets A_k satisfy a sort of independence, namely for a constant $c\geq 1$

$$|A_k\cap A_j|\leq c|A_k||A_j|,\quad k\neq j$$

and then a Borel-Cantelli argument gives that

$$\sum w_n<\infty\qquad\Longrightarrow\quad|\{i.o.A_k\}|=0$$

$$\text{(T)}\qquad\sum w_n=\infty\qquad\Longrightarrow\quad|\{i.o.A_k\}|>0$$

Finally, since $i.o.A_k$ is (eseentially) invariant under rational translations mod 1 the conclusion of (T) improves itself to

$$|\mathbf{R}\setminus\{i.o.A_k\}|=0$$

This argument **works** for **finitely generated Fuchsian groups** whose limit set is $\mathbf{R}$ (the last ingredient is now the ergodicity of the action on $\mathbf{R}$) . Also Jarnik's result holds in this context (Patterson) . But,

Q.6 What about Jarnik-Besicovitch for finitely generated fuchsian groups with R as limit set ?.

Q.7 What about the three results for groups of divergence type ?.

More ambitiously:

Let $\delta = \delta(\Gamma)$ and assume that Γ is of divergence type at δ then there is a Patterson measure which gives full measure to $C(\Gamma)$ and any two such measures are equivalent. Do we have an analogue of Kintchine's theorem or, more interestingly.

Q.8 Is it true that.

$$\mathbf{Dim}(\{\xi : \forall g \in \Gamma, |\xi - g(i)| \geq C_\xi \mu(g)\}) = \delta$$

$$\mathbf{Dim}(\{\xi : \quad for \quad a \quad sequence \quad g_n \in \Gamma |\xi - g_n(i)| \leq C_\xi \mu(g)^t\}) = \delta/t, ift > 1?$$

Higher dimensions . Start from the beginning consider $SL(2, \mathbf{Z} + i\mathbf{Z}) = \Gamma$ one obtains:

$$\left\{z : |z - \tfrac{p}{q}| \leq \tfrac{w_q}{|q^2|} \quad for \quad infinitely \quad many \quad p, q \in \mathbf{Z} + i\mathbf{Z}, ideal \quad (p,q) = \mathbf{Z} + i\mathbf{Z}\right\}$$

have zero or full measure according to

$$\sum w_q^2 < \infty \qquad or \qquad < \infty.$$

Q.9 Is there an analogue of Jarnik-Besicovitch ?.

7. ACTION ON S^{n-1}. MOSTOW'S RIGIDITY THEOREM..

Finally, we describe the action of Γ on S^{n-1}. We need the so-called horospherical limit set A point $\xi \in S^{n-1}$ is a point of horospherical approach for $\Gamma(\xi \in H(\Gamma))$ if $\Gamma(0)$ enters every horosphere base at ξ.

Clearly $C(\Gamma) \subset H(\Gamma) \subset \Lambda(\Gamma)$. Also $H(\Gamma)$ is Γ-invariant. For geometrically finite groups $H(\Gamma) = \Lambda(\Gamma)$

We have:

The action of Γ on $H(\Gamma)$ is conservative.
The action of Γ on $S^{n-1} \setminus H(\Gamma)$ is fully dissipative.
The action of Γ on S^{n-1} $H(\Gamma)$ is fully dissipative iff $H(\Gamma)$ has zero measure.

There is a nice potential-theory characterization due to Pommerenke of those Riemann surfaces for which the action of the covering group on S^1 in fully dissipative. It is easier to state in the case of plane domains. It says that they should be almost of finite type, (Π_1 *is finitely generated*). Let M be a plane domain and Γ its covering group then $|H(\Gamma)| = 0$ iff there exists finitely connected domains $M_n \subset\subset M$ exhausting M so that

$$w(\cdot, \partial M_n \cap \partial M, M) \to 1 \quad as \quad n \to \infty.$$

The potential theory on M is related in some other ways to the action of Γ on S^1

Let M be a Riemann surface with Green's function G. Let Π be the universal cover from Δ onto M. Then

$$G \circ \Pi = log \frac{1}{|B|},$$

where B is the Blaschke product whose zeroes are the orbit of 0 ($\Pi(0)$ = pole of G). Then the fact that Γ is fully dissipative is equivalent to the fact that the derivative of B has finite non-tangential boundary values a.e. on S^1.

The weakest growth condition on a function f analytic in Δ to guarantee that f has finite non-tangential boundary values a.e. on S^1 is to belong to the Nevanlinna class. A Riemann surface is termed **Widom** if B' belongs to N. There is a gorgeons characterization of this concept. It is due to Widom, of course. Fix a pole $p_0 \in M$, and let

$$M(t) = \{z \in M : G(z, p_0) > t\}$$

then $\bigcup_{t>0} M(t) = M$. Let $\beta(t) = \dim H^1(M(t))$ =first Betti number of $M(t)$

Then

M **is Widom iff** $\int_0 \beta(t) < \infty$.

There is a natural conjecture in this issue which I want to introduce. The boundary values of B gives a measurable funtion from S^1 to S^1 which is measure preserving and ergodic (actually, exact). What is his entropy?. Somehow B is like an f-expansion. The conjecture is

B has finite entropy iff M is Widom

More ambitiously, let f be an inner function, $f(0) = 0$, then the entropy of f should be given by

$$h(f) = \frac{1}{2\pi} \int_0^{2\pi} log|f'(e^{i\theta})| d\theta.$$

Mostow's rigidity theorem.

This is the most famous result obtained as a consequence of the ergodic behaviour of Kleinian groups.

In its more essential presentation it claims that if two constant negative curvature compact 3-manifolds are diffeomorphic then they are actually conformally equivalent. (The result is true for complete finite volume n-manifolds of constant negative curvature.)

There is a simple proof of the n-dimensional case due to Ahlfors. But we shall consider only the case $n = 3$. Compare the result with the situation for $n = 2$ where Riemann surfaces can be continuously deformed and with Liouville's theorem asserting that conformal mappings in $\mathbf{R}_n, n \geq 3$, are trivial(=Mobius).

Let $f : M \to M', M, M'$, 3-manifolds as above, and f , diffeomorphism. Since M and M' are compact then f is quasiconformal. Lift f to a quasiconformal map from B^3 onto itself. By classical quasiconformal distortion theorems it can be "restricted" to $\partial B^3 = S^2$, and this restriction is still q.c. and conjugates the boundary action of Γ onto that of Γ'.

Let μ denote the complex dilation of f, i.e. the pull back of the euclidean metric can be writen as $\lambda(z)^2|dz + \mu(z)dz|^2$. (This μ measures the deviation from conformality , $\mu \in L^\infty(S^2)$, $\| \mu \|_\infty < 1$.)

The function μ satisfies

$$\mu(\gamma(z)) = \frac{\gamma'(z)}{|\gamma'(z)|}\mu(z)$$

Thus it determines a vector field on S^2 which is invariant under Γ. The whole point is to show that μ is necessarily $\equiv 0$. Now, we shall see below that this follows since Γ is of divergence type and $n = 3$. But the fact that no such vector field exists on $H(\Gamma)$ (=the conservative part of the action of Γ on S^{n-1}) is true with no assumption on Γ and no restriction on dimension. Returning to the proof in our case. First $|\mu| = cte$, a.e. (divergence type). If $z, w \in S^2, z \neq w$ define $\alpha(z, w) = angle(\mu(z), \mu(w))$ measured along the geodesic joining z, w. Then α is Γ-invariant and so (divergence type) constant, and this is impossible (curvature = +1).

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