ON BANACH SPACES CONTAINING COMPLEMENTED COPIES OF c_0

Giovanni Emmanuele
Dipartimento di Matematica
Università di Catania
Catania 95125,Italy

The following is an exposition of some still unpublished result con=
cerning complemented copies of c_0 in Banach spaces.Our terminology
will be standard.If a capital letter,like E,denotes a Banach space,
then E* denotes its dual space and B(E) its unit ball.

The first result we state needs the concept of limited set.A (boun=
ded) subset X of E is said to be limited ([3]) if for any weak* null
sequence (x_n^*) in E* we have

$$\lim_{n} \sup_{X} |x_n^*(x)| = 0$$

THEOREM 1.For a Banach space E the following are equivalent

(i) E contains a complemented copy of c_0

(ii) there exists a basic sequence (x_n) equivalent to the unit vector
basis of c_0 which is not limited.

Remark.More or less at the same time the previous result was ob=
tained by T.Schlumprecht and it will appear in his Ph.D.Dissertation.

The above ¬uoted result has the three following consequences

THEOREM 2.Let (S,Σ,μ) be a finite measure space and E be a Banach
space containing a copy of c_0.Then the completion $\widehat{\mathcal{P}}(\mu,E)$ of the spa=
ce of Pettis integrable functions with values in E (equipped with the
norm $\| f \| = \sup \left\{ \int_S | x^*f(s)| d\mu : x^* \in B(E^*) \right\}$) contains a complemented
copy of c_0.

This result improves two similar ones ᴉobtained in [4],under certain
restrictions on $\mathcal{P}(\mu,E)$.

THEOREM 3.<u>Let</u> E <u>be a Banach space.Then</u> $l^1(E)$ <u>contains a complemen=</u> <u>ted copy of</u> c_0 <u>iff</u> E <u>does</u>.

We observe that this result cannot be extended to $L^1(\mu,E)$ as it is shown in the paper [5] which will appear on Proc.Amer.Math.Soc.;in passing we note that Prof.Drewnowski observed that the result in [5] can be extended without any trouble to Orlicz function spaces $L^\phi(\mu,E)$.

THEOREM 4.<u>Let</u> E <u>be a Gelfand Phillips space,i.e. a Banach space in</u> <u>which every limited set is relatively compact.If</u> E <u>contains a copy of</u> c_0,<u>it has to contain a complemented copy of</u> c_0.

Using a result from [1] as well as Theorem 4 we can also prove another result (see [1] for the definition of hereditary Dunford Pet= tis property)

THEOREM 5.<u>Let</u> E <u>be a Banach space.Then the following are equivalent</u>

(i) <u>each weakly null sequence</u> $(x_n) \in E$ <u>with</u> $\inf_n \| x_n \| > 0$ <u>has a subsequen=</u> <u>ce</u> $(x_{k(n)})$ <u>equivalent to the unit vector basis of</u> c_0 <u>such that its clo=</u> <u>sed linear span is complemented in</u> E

(ii) E <u>has the hereditary Dunford Pettis property and is a Gelfand Phil=</u> <u>lips space</u>.

Further consequences of the previous results are the following

COROLLARY 6.<u>Let</u> E <u>be adual Banach space complemented in a Banach</u> <u>lattice.</u>E <u>is a Gelfand Phillips space iff</u> $E \not\supset c_0$.

We recall that if E has l.u.st. ([6]) or G.L.l.u.st. ([7]) then E* is complemented in a Banach lattice.

In the following corollary Σ will be an algebra of subset of an ab= stract set S and $fabv(\Sigma,E)$ will denote the usual Banach space of E-va=

lued vector measures

COROLLARY 7.Let E* be complemented in a Banach lattice.Then fabv(Σ,E*)

is a Gelfand phillips space iff E* is.

The final results are consequences of the previous theorems;they

concern with operators defined on suitable Banach spaces

COROLLARY 8.Let E be a Banach lattice.The following are equivalent

(i) E is a Grothendieck space (see [2])

(ii) any operator T:E$\longrightarrow$ F,F a Gelfand Phillips space,is weakly com=

pact

(iii) any operator T:E$\longrightarrow$ c_0 is weakly compact.

COROLLARY 9.Let E be a Banach space with the property (V) (see[8]).

Then any strict cosingular (see [9]) operator T:E$\longrightarrow$ F,F a Gelfand

Phillips space,is weakly compact.

REFERENCES

1. P. CEMBRANOS. The hereditary Dunford-Pettis property on C(K,E). Illinois. J. Math. 31(1987), 365-373.

2. J. DIESTEL; J.J. UHL, Jr. Vector Measures. Math. Surveys 15 AMS 1977.

3. J. DIESTEL; J.J. UHL, Jr. Progress in Vector Measures. 1977-83. Lectu re Notes in Math. 1033. Springer Verlag. 1984.

4. P. DOWLING, G. EMMANUELE; R. RYAN. On complemented copies of c_0 in Banach spaces of operators and H^p-vector valued functions, to appear.

5. G. EMMANUELE. On complemented copies of c_0 in L_X^p, $1 \le p < \infty$ to appear.

6. T. FIGIEL; W. JOHNSON; L. TZAFRIRI. On Banach lattices and spaces having local unconditional structure, with applications to Lorentz spaces. J. Approx. Theory 13(1975) 395-412.

7. y. GORDON; D. LEWIS. Absolutely summing operators and local unconditional structure. Acta Math. 133(1974) 27-48.

8. A. PELCZYNSKI. Banach spaces on which every unconditional converging operator is weakly compact. Bull. Acad. Pol. Sci. 10(1962) 641-648.

9. A. PELCZYNSKI. On strictly singular and strictly cosingular operators Bull. Acad. Pol. Sci. 13(1965) 31-41.