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# CHARACTERIZATIONS OF TAUBERIAN OPERATORS AND OTHER SEMIGROUPS

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Tauberian operators, studied by Kalton and Wilansky [9], appear in different situations: summability [3], factorization of operators [1], [11], preservation of isomorphic properties of Banach spaces [11], equivalence between Radon-Nikodym property and Krein-Milman property [15], and generalized Fredholm operators [17], [18]. Recently Neidinger and Rosenthal [12] have obtained characterizations of tauberian operators in terms of the closedness of images of closed sets.

In this paper we present characterizations of tauberian and cotauberian (conjugate tauberian [17]) operators, and of other semigroups of operators studied in [4], [5], formally analogous in some sense to the class of tauberian operators.

For an operator ideal V, the space ideal Sp(V) is the class of all Banach spaces X whose identity $I_X$ is in V, the dual operator ideal $V^d$ is the class of operators K such that K' is in V, and $Sp(V^d) = \{ X / X' \in Sp(V) \}$ [13].

Let U be one of the operator ideals Co, WCo, Ro, CC, and WCC of all compact, weakly compact, Rosenthal, completely continuous and weakly completely continuous operators respectively. Note that Sp(U) is respectively the class of all finite dimensional, reflexive, without copies of $l_1$, Schur or weakly sequentially complete Banach spaces.

In [4] and [5] we defined two semigroups $SU_+$ and $SU_-$ of operators such that $SCo_+$ and $SCo_-$ coincide with the classes $SF_+$ and $SF_-$ of upper and lower semi-Fredholm operators and they are contained in $SU_+$ and $SU_-$ respectively.

L(X,Y) will denote the set of all operators from X into Y; and $T' \in L(Y',X')$ the conjugate operator of $T \in L(X,Y)$.

<u>Theorem</u> 1 Suppose $U \in \{ Co, WCo, CC, Ro, WCC \}$ and $T \in L(X,Y)$.
(a) $T$ is in $SU_+$ if and only if $N(T+K)$ belongs to $Sp(U)$ for every compact operator $K$.
(b) $T$ is in $SU_-$ if and only if $Y/\overline{R(T+K)}$ belongs to $Sp(U^d)$ for every compact operator $K$.

<u>Observation</u> 2 The operator ideals considered in the paper are related as follows:

$$Co \subset WCo \cap CC, \quad WCo \subset Ro \cap WCC, \quad CC \subset WCC$$

From theorem 1 it follows that the same inclusions are true for the corresponding semigroups $SU_+$ (or $SU_-$ ).

Given a (closed) subspace $M$ of $X$, $i_M$ will denote the inclusion of $M$ into $X$, and $q_M$ the quotient map onto $X/M$.

<u>Theorem</u> 3 Suppose $U \in \{ Co, WCo, CC, Ro, WCC \}$ and $T \in L(X,Y)$.
(a) $T \in SU_+$ if and only if for every Banach space $Z$ and $A \in L(Z,X)$ we have that $TA \in U$ implies $A \in U$; equivalently for every subspace $M$ of $X$, $Ti_M \in U$ implies $M \in Sp(U)$.
(b) $T \in SU_-$ if and only if for every Banach space $Z$ and $B \in L(Y,Z)$ we have that $BT \in U^d$ implies $B \in U^d$; equivalently for every subspace $N$ of $Y$, $q_N T \in U^d$ implies $Y/N \in Sp(U^d)$.

<u>Observation</u> 4 In [5] we considered two other operator ideals $Gr$ and $Cd$ defined in terms of the weak$^*$ convergence in dual spaces.

The proof of part (b) in the above theorem can be adapted to the corresponding semigroups $SGr_-$ and $SCd_-$.

<u>Proposition</u> 5 Let $T \in L(X,Y)$.
(a) If $T \in SU_+$ and $Y \in Sp(U)$, then $X \in Sp(U)$.
(b) If $T \in SU_-$ and $X \in Sp(U^d)$, then $Y \in Sp(U^d)$.

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