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UNCONDITIONALLY CONVERGING OPERATORS AND FREDHOLM THEORY

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In [7] tauberian operators were studied as a class of operators with opposite properties to that of the weakly compact operators. Recently we have introduced in [3], [4] classes of operators associated with the (weakly) completely continuous and l_1-singular operators and its conjugates in analogous way as tauberian operators are associated with weakly compact operators. We were able to extend the results of [7] to these new classes, using essentially lifting results for bounded, weakly Cauchy and (weakly) convergent sequences [5] similar to that of Lohman [8].

In this paper, by using results of Bessaga and Pelczynski about basic sequences and unconditionally converging (UC) operators [1], [10], [11], we study two new classes of operators SUC_+ and SUC_- associated with the UC operators. We prove a lifting result for unconditionally converging series which allow us to extend some results of [3], [4], [7] to this situation. Specifically, we characterize semi-Fredholm operators, somewhat-c_o Banach spaces (each infinite dimensional subspace contains c_o) and quowhat-l_1 Banach spaces (each infinite dimensional quotient has a quotient isomorphic to l_1) in terms of the two operator classes here introduced.

$T \in L(X,Y)$ is upper (lower) semi-Fredholm if the range $R(T)$ is closed and the kernel $N(T)$ (cokernel $Y/\overline{R(T)}$) is finite dimensional.

A series Σx_n in a Banach space X is weakly unconditionally Cauchy (w.u.C.) if $\Sigma |f(x_n)| < \infty$ for $f \in X'$.

$K \in L(X,Y)$ is unconditionally converging if it sends w.u.C. series in X into unconditionally convergent series in Y [10].

$T \in L(X,Y)$ belongs to SUC_+ if given a w.u.C. series Σx_n in X, ΣTx_n unconditionally convergent implies Σx_n unconditionally convergent.

T belongs to SUC_- if the conjugate $T' \in SU_+$.

It is clear from the definition that SUC_+ and SUC_- are semigroups (stable by product). Moreover we have the following results.

Proposition 2 Let $T, K \in L(X,Y)$.
(a) If $T \in SUC_+$ and $K \in UC$, then the kernel $N(T)$ contains no copies of c_o and $T + K \in SUC_+$.
(b) If $T \in SUC_-$ and $K' \in UC$, then the cokernel $Y/\overline{R(T)}$ contains no complemented copies of l_1 and $T + K \in SUC_-$.

We observe that SUC_+ is a class of generalized upper semi-Fredholm operators for which the unconditionally converging operators and the spaces with no copies of c_o play the role of the compact operators and the finite dimensional spaces of the classic theory. A similar remark is in order for SUC_-.

Proposition 3 Let X, Y be Banach spaces.
(a) If Y contains no copies of c_o and there exists an operator T in $SUC_+(X,Y)$, then X contains no copies of c_o.
(b) If X contains no complemented copies of l_1 and there exists an operator T in $SUC_-(X,Y)$, then Y contains no complemented copies of l_1.

Theorem 4 Let X be a Banach space, M a subspace of X containing no copies of c_o, and q the quotient map onto X/M. If Σx_n is a w.u.C. series in X such that Σqx_n is convergent, then Σx_n is convergent.

Given a subspace M of a Banach space X we denote by i_M the inclusion of M, and by q_M the quotient map onto X/M.

Theorem 5 Let $T \in L(X,Y)$.

(a) T is upper semi-Fredholm if and only if $T \in SUC_+$ and for every subspace M of X containing no copies of c_o, the restriction Ti_M is upper semi-Fredholm.

(b) T is lower semi-Fredholm if and only if $T \in SUC_-$ and for every subspace N of Y with Y/N containing no complemented copies of l_1 the operator q_NT is lower semi-Fredholm.

A Banach space Y is somewhat-c_o if every infinite dimensional subspace of Y contains a copy of c_o; and Y is quowhat-l_1 if every infinite dimensional quotient of Y has a quotient isomorphic to l_1.

Theorem 6 Let Y be a Banach space.

(a) Y is somewhat-c_o if and only if for every Banach space Z each operator in $SUC_+(Y,Z)$ is upper semi-Fredholm.

(b) Y is quowhat-l_1 if and only if for every Banach space X each operator in $SUC_-(X,Y)$ is lower semi-Fredholm.

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