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Almost Everywhere Convergence of Fourier Integrals in $\mathbb{R}^n$.

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For suitable f, defined on $\mathbb{R}^n$, $n \geq 2$, we let $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x.\xi}\, dx$, and

$S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi)\, e^{2\pi i x.\xi}\, d\xi$. In this note we wish to announce some results concerning classes of functions for which almost everywhere convergence of $S_R f$ holds, i.e.

$$\lim_{R \to \infty} S_R f(x) = f(x) \quad \text{a.e.} \tag{1}$$

We also consider the lacunary problem

$$\lim_{k \to \infty} S_{2^k} f(x) = f(x) \quad \text{a.e.} \tag{2}$$

Classical results on the unit circle T include (with S_k the k'th partial sum operator)

i) (2) fails for some f in $L^1(T)$ (Kolmogorov)

ii) If $f \in L^1(T)$ vanishes in some open set Ω, then (1) holds on Ω (Riemann localisation principle)

iii) (1) holds uniformly in x if f is continuous and (w(t) denoting the L^∞ modulus of continuity) $\int_0^1 w(t)\, dt/t < \infty$ (Dini test)

iv) Uniform convergence of (1) does not hold in general for continuous functions (du Bois Reymond)

v) (1) holds for $f \in L^p(T)$ if $1 < p \leq \infty$ (Carleson [2], Hunt[4])

<u>Theorem 1.</u> If $f \in L^p(\mathbb{R}^n)$, with $2 \leq p < 2n/(n-1)$, $n \geq 2$, and $f \equiv 0$ on an open set Ω of $\mathbb{R}^n$, then (1) holds a.e. on Ω

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__Remark__. This range of p is the best we can hope localisation to hold for: Fefferman's theorem [3] shows it cannot hold for $p<2$ (since $n\geq 2$), and elementary considerations show that $p<2n/(n-1)$ is necessary. An earlier variant of Theorem 1 where f was assumed to have compact support was obtained by Sjölin [5].

Theorem 1 is a consequence of the following weighted L^2 estimate: Let K_R be the convolution kernel of S_R, and let $T_R{}^j$ be the operator whose kernel is $K_R(x)\ \psi(x/2^j)$, where ψ is a smooth function of compact support satisfying $\sum_{j\geq 1}\psi(x/2^j)\equiv 1$ in the set $\{x : |x|\geq 2\}$

__Theorem 2.__ Let $0\leq\beta<n/2$ and $\gamma > \max(\beta-1/2,\ 0)$. Then,

$$\int_{\mathbb{R}^n} \sup_{R>1} \left|\sum_{j\geq 1} T_R{}^j f_j(x)\right|^2 dx/|x|^{2\beta} \leq C_{\beta,\gamma} \sum_{j\geq 1} 2^{2\gamma j} \int_{\mathbb{R}^n} |f_j|^2\ dx/|x|^{2\beta}$$

The techniques used to prove Theorem 2 derive in part from [1] and ultimately from [6]. It is worth noting that, in [1], the following theorem is also proved

__Theorem A.__ If $2 \leq p < 2n/(n-1)$, then (2) holds for $f\in L^p(\mathbb{R}^n)$.

We now turn to see how the situation may improve if we assume some smoothness of f.

__Theorem 3.__ If $\int |\hat{f}(\xi)|^2\ (1+\log^+|\xi|)^2\ d\xi < \infty$, then (1) holds.

For $1 \leq p < \infty$, and $\alpha>0$, let $L_\alpha{}^p(\mathbb{R}^n) = \{ f : [(1+|\xi|^2)^{\alpha/2}\ \hat{f}(\xi)] \in L^p(\mathbb{R}^n) \}$ be the Sobolev space of functions on $\mathbb{R}^n$ whose derivatives up to order α are in $L^p(\mathbb{R}^n)$. Theorem 3 clearly implies the case $p = 2$ of the following theorem

__Theorem 4.__ If $f\in L_\alpha{}^p(\mathbb{R}^n)$, with $2 \leq p < 2n/(n-1)$, $\alpha >0$, then (1) holds.

To conclude the discussion of the case $p \geq 2$, we state Theorem 5, which may regarded as an analogue for $\mathbb{R}^n$ of Dini's test

Theorem 5. Let $n \geq 2$. If $f \in L_{(n-1)/2}^{2n/(n-1),1}(\mathbb{R}^n)$, then $S_R f$ converges to f uniformly.

(Here $L_\alpha^{p,1}(\mathbb{R}^n)$ is the space of functions whose derivatives up to order α are in the Lorentz space $L^{p,1}$.) For $p < 2$, much less is known. A surprising result (perhaps) is

Theorem 6. Let $n \geq 2$. If $f \in L_{(n-1)/2}^1(\mathbb{R}^n)$, then (1) holds.

Notice that Kolmogorov's example shows this theorem fails when $n=1$. We also have

Theorem 7. If $n=2$ and $\alpha>0$ or if $n\geq3$ and $\alpha>(n-1)/2(n+1)$, (2) holds for $f \in L_\alpha^p(\mathbb{R}^n)$ whenever $2n/(n+1+2\alpha) < p \leq 2$.

The reader will notice that Theorem 7 holds precisely for those α, p and n for which the Bochner-Riesz multipliers $(1-|\xi|^2)_+^\alpha$ are known to give bounded operators on $L^p(\mathbb{R}^n)$; in fact, a close connection with Bochner-Riesz theory characterizes all stages of this work.

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