EXTRACTA MATHEMATICAE 3, n.1, 40-42 (1988)

DIMENSION OF DENSE SUBALGEBRAS OF C(X)

Juan B. Sancho de Salas

Mª Teresa Sancho de Salas

AMS 1980 Subject Classification: 54C40,54F45.

The aim of this work is to determine the dimension of any compact Hausdorff space X in terms of the Krull dimension of dense subalgebras of C(X).

In Algebraic Geometry, the dimension of an affine algebraic variety V is defined as the Krull dimension of the ring A of all algebraic functions on V (i.e., the affine coordinate ring, so that $V = \operatorname{Spec} A$):

$$\dim V = \dim A = \begin{bmatrix} \text{Supremum of the lengths of all} \\ \text{chains of prime ideals in A} \end{bmatrix} \qquad (*)$$

Moreover, Noether's Lemma states that $\dim V \leq n$ if and only if there exists a finite morphism from V to the affine n-space (i.e., a finite morphism from a polynomial ring $k[x_1,\ldots,x_n]$ to A).

In Topology, the equivalent result to Noether's Lemma is simply Katětov's characterization of the dimension: A compact metric space X has dimension $\leq n$ if and only if there exists a continuous map $X \longrightarrow \mathbb{R}^n$ with totally disconnected fibers. In terms of C(X), we have $\dim X \leq n$ if and only if there exists a morphism from a polynomial ring $\mathbb{R}[x_1,\ldots,x_n]$ to C(X) such that the image is an analytic base for C(X).

Our main result, which may be considered as analogous to (*) in Topology, is the following theorem:

THEOREM: A compact metric space has dimension $\leq n$ if and only if it is the real spectrum of an algebra of dimension $\leq n$.

In other words, the dimension of any compact metric space X is the minimum of the Krull dimensions of all dense subalgebras of C(X).

Let X be a topological space and let A(X) be the family of all closed subsets. If we consider on A(X) the addition and the product defined by the intersection and the union respectively, then A(X) is a distributive lattice. We say that a sublattice B of A(X) is a basis for the topology of X if any closed set is an intersection of closed sets in B. The arithmetical dimension of X is the minimum of all integers n such that there exists a basis for X of dimension n and we denote it by $\dim X$. From now on, the arithmetical dimension of a space X will be referred to simply as the dimension of X.

R. Galián ([4]) proved that **the arithmetical dimension is monotone and, for separable metric spaces, it coincides with the inductive dimension and the cover dimension.** For compact Hausdorff spaces, we have the following inequalities ([6]) between the different dimension functions:

$$\text{cover dim } X \leq \text{ind } X \leq \text{Ind } X \leq \dim X$$

and there exists an example (Filippov[2]) of a compact Hausdorff sapce X such that cover dim $X=1$, ind $X=2$ and Ind $X=3$. Moreover, there exists (Filippov [3]) a pair X,Z of compact Hausdorff spaces such that Ind $X=1$, Ind $Z=2$ and Ind$(X\times Z)\geq 4$. Since Galián ([4]) proved that $\dim(X\times Z)\leq \dim X+\dim Z$, we conclude that Ind $K<\dim K$ for some compact Hausdorff space K.

Let A be an $\mathbb{R}$-algebra. We say that a maximal ideal M of A is real if the canonical map $\mathbb{R}\longrightarrow A/M$ is an isomorphism. The real spectrum of A is the set $\mathrm{Spec}_{\mathbb{R}} A$ of all real maximal ideals of A. Given a point x of $\mathrm{Spec}_{\mathbb{R}} A$, the corresponding maximal ideal of A will be denoted by M_x. Note that each element $f\in A$ defines a real function on $\mathrm{Spec}_{\mathbb{R}} A$ by setting: $f(x)$ = residue class of f in $A/M_x=\mathbb{R}$. We shall consider on $\mathrm{Spec}_{\mathbb{R}} A$ the initial (or weak) topology for these functions.

Let X be a topological space and let A be a subalgebra of the ring C(X) of all real-valued continuous functions on X. We say that A separates points if for any two points $x,y\in X$ there exists $f\in A$ such that $f(x)\neq f(y)$, and we say that A is basic if X has the initial topology for the functions in the family A. When X is a compact Hausdorff space, Stone-Weierstrass Theorem states that A is basic if and only if it is dense in C(X).

THEOREM: $\dim(\mathrm{Spec}_{\mathbb{R}} A)\leq \dim A$, for any $\mathbb{R}$-algebra A.

Corollary: If A is a basic algebra of continuous functions on a topological space X, then $\dim X\leq\dim A$.

By Freudenthal's Theorem ([1],1.13.2), any compact metric space of dimension $\leq n$ is an inverse limit of finite triangulated polyhedra of dimension $\leq n$. Using it we may construct a basic algebra A of dimension $\leq n$, and we get:

THEOREM: The dimension of any compact metric space X is the minimum of the dimensions of all dense subalgebras of C(X).

The extension of this theorem to arbitrary compact Hausdorff spaces would be a direct consequence of a positive answer to the following question:

Let X be a compact Hausdorff space of dimension n. Is X a subspace of an inverse limit of n-dimensional finite triangulated polyhedra and semilinear maps?

If X is an inverse limit of n-dimensional finite triangulated polyhedra and semilinear maps, we know that there exists a basic algebra of continuous functions on X of dimension $\leq n$. Hence the answer to the question is negative when the arithmetical dimension dim is replaced by cover dim, ind or Ind.

Two facts suggest that the answer may be affirmative. First, Isbell's Theorem ([6]) stating that any n-dimensional T_o-space is a subspace of an inverse limit of finite n-dimensional spaces. Furthermore, each finite space gives rise to a finite triangulated polyhedron of the same dimension (named the geometric realization of the given finite space).

Hence we hope that our characterization may be extended to arbitrary compact Hausdorff spaces when one consider the arithmetical dimension instead of the cover dimension or the inductive dimension. Anyway, we obtain the following characterization for the covering dimension of arbitrary compact Hausdorff spaces:

THEOREM: A compact Hausdorff space X has cover dimension $\leq n$ if and only if every countably generated subalgebra of C(X) is contained in the closure of a subalgebra of Krull dimension $\leq n$.

REFERENCES

[1] Engelking, R.: Dimension Theory, North-Holland Math. Library nº 19, Amsterdam (1978).

[2] Filippov, V.V.: On bicompacta with noncoinciding inductive dimensions, Dokl. Akad. Nauk SSSR 192 (1970), 289-292.

[3] Filippov, V.V.: On the inductive dimension of the product of bicompacta, Dokl. Akad. Nauk SSSR 202 (1972), 1016-1019.

[4] Galián, R.: Teoría de la dimensión, Serie Univ. de la Fund. Juan March nº 107, Madrid (1979).

[5] Gillman, L. and Jerison, M.: Rings of continuous functions, Graduate Texts in Math. nº 43, Springer-Verlag, Heidelberg (1976).

[6] Isbell, J.: Graduation and dimension in locales, London Math. Soc. Lecture Notes Ser. nº 93, Cambridge Univ. Press, Cambridge (1985), 195-210.

Juan B. Sancho de Salas
Dpto. de Matemáticas
Univ. de Extremadura
Avda. de Elvas s/n
06071-Badajoz
(Spain)

Mª Teresa Sancho de Salas
Dpto. de Matemáticas
Univ. de Salamanca
Pza. de la Merced 1-4
37008-Salamanca
(Spain)

(To appear in Proceedings of the Amer. Math. Soc.)