EXTRACTA MATHEMATICAE 3, n.1, 37-39 (1988)

SOME RESULTS ON DIMENSION THEORY: UNIVERSAL SPACES.

Juan B. Sancho de Salas
Mª Teresa Sancho de Salas

Given an infinite cardinal w and a non-negative integer , we prove the existence of a universal space $X_{w,n}$ of weight w and dimension n such that any space of weight $\leq w$ and dimension $\leq n$ is homeomorphic to a subspace of $X_{w,n}$.

The dimension function we use was proposed by J.B. Sancho Guimerá and developed by R. Galián in [2] for topological spaces. This dimension coincides with the inductive dimension for separable metric spaces, and with Grothendieck's combinatorial dimension ([3]) for noetherian spaces (see [2] or [6]).

J. Isbell ([4]) introduced an equivalent dimension function for locales, named the graded dimension. By definition, a space has graded dimension n if and only if it is homeomorphic to a subspace of an inverse limit of finite spaces of dimension $\leq$ n. This property will be essential in this work.

Any set A with two binary composition laws (addition and product) is said to be a **lattice** if it is a commutative semigroup under both operations, the product is distributive over the addition, and for any $a \in A$ we have: $a\cdot 0=0$, $a^2=a$, $a+1=1$.

A map $f:A \longrightarrow B$ between two lattices is said to be a **lattice morphism** if it preserves the addition and the product, and $f(0)=0$, $f(1)=1$.

A subset B of a lattice A is said to be a **sublattice** if it closed under addition and product and $0,1 \in B$.

A non-empty subset I of a lattice A is said to be an **ideal** if it closed under addition and stable under product by arbitrary elements of A. An ideal P is **prime** if it is not A and, whenever the product of two elements belongs to P, then at least one factor belongs to P.

The (Krull) **dimension** of a lattice A is the supremum of all integers n such that there exists a chain $P_o \subset P_1 \subset \dots \subset P_n$ of prime ideals in A, and we denote it by $\dim A$.

If X is a topological T_o-space, we denote by A(X) the lattice of all closed subsets of X, where the addition and product are the intersection and union respectively. A sublattice B of A(X) is said to be a **basis** for X if any closed set in X is an intersection (eventually infinite) of elements of B. The (arithmetical) **dimension** of X is defined to be the minimum of the dimensions of all bases for X and we denote it by $\dim X$.

The following Theorem is essentially a result of J. Isbell ([4], 2.6). He proves that the graded dimension coincides with the Krull dimension for spectral spaces. As can easely be seen, this result implies that the arithmetical and graded dimension are equivalent, and it may be restated as follows:

THEOREM: Any n-dimensional space is a subspace of an inverse limit of finite spaces of dimension $\leq$ n.

The **weight** of a T_0-space is defined to be the minimum of the cardinalities of all bases for it. Our main result is the following theorem:

EXISTENCE OF UNIVERSAL SPACES: There exists an n-dimensional space $X_{w,n}$ of weight w such that any space of weight $\leq$ w and dimension $\leq$ n is homeomorphic to a subspace of $X_{w,n}$.

The proof of this theorem is based on the following results:

FACTORIZATION THEOREM: Let $f: A \longrightarrow B$ be a lattice morphism. Then there exists a sublattice C of B which contains the image of f and satisfies the following properties:
1) $\dim C \leq \dim B$.
2) If A is finite, then so is C. If A is infinite, then $\operatorname{card}(C) \leq \operatorname{card}(A)$.

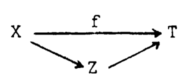
THEOREM: If a lattice A is a direct product of n-dimensional lattices, then the dimension of A is n.

These results may be used to get consequences for compact Hausdorff spaces:

THEOREM: There exists a n-dimensional compact Hausdorff space $K_{w,n}$ of weight w, such that any compact Hausdorff space of weight $\leq$ w and dimension $\leq$ n is homeomorphic to a subspace of $K_{w,n}$.

The existence of Universal spaces for the covering dimension is based on Mardešić's Factorization Theorem (see [1]). Even though we don't use it when proving the above results, there is also a Factorization Theorem for the arithmetical dimension and compact Hausdorff spaces:

FACTORIZATION THEOREM: Let $f: X \longrightarrow T$ be a continuous map between compact Hausdorff spaces. If the dimension of X is n and the weight of T is w, then f factors through a compact Hausdorff space Z of weight $\leq$ w and dimension $\leq$ n:



REFERENCES

[1] R. Engelking: **Dimension Theory**, North-Holland Math. Library No. 19, Amsterdam (1978).

[2] R. Galián: **Teoría de la dimensión**, Serie Univ. de la Fund. Juan March No. 107, Madrid (1979).

[3] A. Grothendieck: **E.G.A. IV** (Première partie), Publ. Mat. I.H.E.S. No. 20, Paris (1964).

[4] J. Isbell: **Graduation and dimension in locales**, London Math. Soc. Lecture Notes Ser. No. 93, Cambridge Univ. Press, Cambridge (1985), pp. 195-210.

[5] P.T. Johnstone: **Stone spaces**, Cambridge studies in adv. Math. No. 3, Cambridge Univ. Press, Cambridge (1982).

[6] M.T. Sancho: **Methods of Commutative Algebra for Topology**, Publ. Mat. No. 17, Univ. de Extremadura, Badajoz (1987).

Juan B. Sancho de Salas
Dpto. de Matemáticas
Univ. de Extremadura
Avda. de Elvas s/n
06071 - Badajoz
(Spain)

Mª Teresa Sancho de Salas
Dpto. de Matemáticas
Univ. de Salamanca
Pza. de la Merced 1-4
37008 - Salamanca
(Spain)

(These results were presented at the International Conference on Categories held at Louvain-la-Neuve (Belgium) in the Summer of 1987)