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ORTHOGONALITY IN NORMED LINEAR SPACES: A SURVEY
PART I: MAIN PROPERTIES

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Let E be a **real linear space**. An **inner product** in E is a mapping $(\ |\ ) : E\times E \longrightarrow \mathbb{R}$ such that

$$(\lambda x+\mu y|z)=\lambda(x|z)+\mu(y|z) \quad , \quad (x|y)=(y|x) \quad , \quad (x|x)>0 \text{ if } x\neq 0$$

for every $\lambda,\mu\in\mathbb{R}$ and $x,y\in E$.

A **norm** in E is a mapping $\|\ \| : E \longrightarrow \mathbb{R}$ such that

$$\|x+y\|\leq\|x\|+\|y\| \quad , \quad \|\lambda x\|=|\lambda|\,\|x\| \quad , \quad \|x\|>0 \text{ if } x\neq 0$$

for every $\lambda\in\mathbb{R}$ and $x,y\in E$.

Every inner product induces a norm $\|x\|=(x|x)^{1/2}$ satisfying the **parallelogram law**

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2) \quad , \quad (x,y\in E)$$

Reciprocally every norm satisfying the parallelogram law is induced by the inner product [22]

$$(x|y)=\frac{1}{4}(\|x+y\|^2-\|x-y\|^2) \quad , \quad (x,y\in E)$$

Therefore a real normed linear space is an inner product space (its norm is induced by an inner product) if and only if so is every two-dimensional linear subspace (briefly a **plane**) of it.

The plane generated by two linearly independent points x and y of a real normed linear space E can be identified with the linear space $\mathbb{R}^2$ endowed with the norm $\|(\lambda,\mu)\|=\|\lambda x+\mu y\|$. With this assumption we can say that E is an inner product space if and only if every plane section of its **unit sphere** $S=\{x\in E : \|x\|=1\}$ is an ellipse.

Let E be an **inner product space (i.p.s.)**. A point $x \in E$ is said to be **orthogonal** to other point $y \in E$, $x \perp y$, when $(x|y)=0$.

Orthogonality relation is well equipped with interesting properties. The way in which we describe below the more significative of such properties is slightly reiterative but suitable for our later study.

MAIN PROPERTIES OF ORTHOGONALITY IN INNER PRODUCT SPACES

Nondegeneracy: $\lambda x \perp \mu x$ if and only if either $\lambda x=0$ or $\mu x=0$.

Simplification: If $x \perp y$ then $\lambda x \perp \lambda y$, $(\lambda \in \mathbb{R})$.

Continuity: If $x_n \perp y_n$ for every $n \in \mathbb{N}$ and if the sequences (x_n) and (y_n) converge to x and y, respectively, then $x \perp y$.

Homogeneity: If $x \perp y$ then $\lambda x \perp \mu y$, $(\lambda, \mu \in \mathbb{R})$.

Symmetry: If $x \perp y$ then $y \perp x$.

Additivity: If $x \perp y$ and $x \perp z$ then $x \perp y+z$.

Existence: For every oriented plane **P**, every $x \in P \setminus \{0\}$ and every $\rho > 0$, there exists $y \in P$ such that the pair $[x,y]$ is in the given orientation, $\|y\|=\rho$ and $x \perp y$.

Uniqueness: The above y is unique.

Existence unique diagonals: For every $x, y \in E \setminus \{0\}$ there exists a unique $\rho > 0$ such that $x+\rho y \perp x-\rho y$. (i.e. among every parallelograms with sides in given directions there are some with orthogonal diagonals. Moreover such parallelogram is unique if we fix an orientation on the plane generated by the sides and the norm of one side.)

On the other hand there are many ways to word orthogonality of two points without explicit mention to the inner product of the space. Among them we pay attention to the following ones.

PROPOSITIONS EQUIVALENT TO THE FACT X⊥Y

(R) $\|x+\lambda y\|=\|x-\lambda y\|$, $(\lambda\in\mathbb{R})$

(B) $\|x\|\leq\|x+\lambda y\|$, $(\lambda\in\mathbb{R})$

(I) $\|x+y\|=\|x-y\|$

(P) $\|x-y\|^2=\|x\|^2+\|y\|^2$

(S) Either x=0 , or y=0 , or $\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|=\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|$

(C) $\sum_{i=1}^{m}\alpha_i\|\beta_i x+\gamma_i y\|^2=0$, where $\alpha_i,\beta_i,\gamma_i$ are real numbers such that

$$\sum_{i=1}^{m}\alpha_i\beta_i^2=\sum_{i=1}^{m}\alpha_i\gamma_i^2=0 \quad , \quad \sum_{i=1}^{m}\alpha_i\beta_i\gamma_i=1$$

(D) $\sup\{f(x)g(y)-f(y)g(x)\ :\ f,g\in S'\}=\|x\|\,\|y\|$, where S' denotes the unit sphere of the dual space E'.

(A) Either x=0, or y=0, or they are linearly independent and such that the four sectors defined by x and y in the unit ball of the plane generated by them (identified to $\mathbb{R}^2$) are of the same area.

Propositions (I) and (P) are particular cases of (C), as well as other propositions considered by some autors in the same context (orthogonality in normed linear spaces) of the present survey. For example

$\alpha,\beta\in(0,1)$ and $\|x+y\|^2+\|\alpha x+\beta y\|^2=\|\alpha x+y\|^2+\|x+\beta y\|^2$ [24]

$\alpha\neq 1$ and $(1+\alpha^2)\|x+y\|^2=\|x+\alpha y\|^2+\|y+\alpha x\|^2$ [15]

$\lambda\neq 0$ and $\|x+\lambda y\|=\|x-\lambda y\|$ [9] [3]

$\lambda\neq 0$ and $\|x-\lambda y\|^2=\|x\|^2+\lambda^2\|y\|^2$ [9] [3]

As we have indicated above anyone of the mentioned propositions is equivalent to the orthogonality of the points x and y. On the other hand they are meaningful also in normed linear spaces. Both facts are in the basis of every concept of **orthogonality in normed linear spaces** which we shall consider henceforth.

ORTHOGONALITY IN NORMED LINEAR SPACES

Let E be a **real normed linear space.** Following the chronological order of the respective concepts we shall say that a point $x \in E$ is **orthogonal** to other point $y \in E$ **in the Roberts sense** [27,(1934)], $x \perp^{R} y$, when they satisfy the proposition (R). Analogously for the senses: **Birkhoff** [7,(1935)], **Isosceles** [18 ,(1947)], **Pytagorean** [18 ,(1947)], **Singer** [28,(1957)], **Carlsson** [10,(1961)], **Diminnie** [14,(1983)] and **Area** [2,(1984)].

There are two primary and natural family of questions relative to the several types of orthogonality in normed linear spaces:

First, what properties of orthogonality in inner product spaces (symmetry, homogeneity,...) remain valid for this generalized conceps.

Second, what is the relation (equivalence, one implies other, ...) between any two orthogonalities in a given normed linear space.

From the first family of questions there is only a little part that is common to every orthogonalities under consideration, namely the **nondegeneracy, simplification** and **continuity.** All of them easy to prove.

With regard to the other basic properties listed before the results are very assorted and occasionally difficult to prove or unknown. The part I of this survey is devoted to a description of many known results on this topic.

MAIN PROPERTIES OF R-ORTHOGONALITY

Besides the already mentioned for every generalized orthogonality (nondegeneracy, simplification, and continuity), it is obvious that **R-orthogonality is homogeneous and symmetric.**

Only with a little intuition on the geometrical meaning of R-orthogonality it is easy to achieve examples (e.g. in $\mathbb{R}^3$) of spaces in which **R-orthogonality is non additive.**

With regard to the **existence** R.C. James [18] gave an example of space in which two points are R-orthogonal if and only if any of them is zero. (R-orthogonality is trivially additive in such a space).

Furthermore James proved in the mentioned paper that

R-orthogonality is existing if and only if the norm is induced by an inner product.

In other words the interest of this concept of orthogonality just ends in this remarkable characteristic property of inner product spaces, which was obtained as a corollary of the F.A. Ficken's theorem [16]:

"E is an inner product space if and only if $\|x+\alpha y\|=\|x-\alpha y\|$, for every $\alpha\in\mathbb{R}$ and $x,y\in E$".

For analogous reasons also the **admission of diagonals for this orthogonality is characteristic of inner product spaces.**

Finally it is easy to prove the **uniqueness and uniqueness of diagonals, in case they exist.** (A fortiori, the trivial additivity in spaces of dimension two.)

MAIN PROPERTIES OF B-ORTHOGONALITY

It is obvious that **B-orthogonality is homogeneous.**

However **it is, in general, neither symmetric nor additive.**

G. Birkhoff [7], R.C. James [19,20] and M.M. Day [11] in gradual stages proved that:

A real normed linear space of dimension ≥ 3 is an inner product space if and only if B-orthogonality is symmetric.

This outstanding result is closely related with some deep and involved theorems of W. Blaschke [8], S. Kakutani [23], R. Phillips [26] and others. Among several equivalent statements of such theorems we single out the following ones:

"The unit sphere S of a norm in $\mathbb{R}^3$ is an ellipsoid if and only if the set of contact points with S of any cillinder circunscribed to S contains a plane section of S." [25]

"A real normed linear space of dimension ≥ 3 is an inner product space if and only if there is a linear projection of norm 1 over each plane." [23] [26]

With regard to the **two-dimensional case** G. Birkhoff [7] indicated the way to construct examples of norms in $\mathbb{R}^2$ with symmetric B-orthogonality but non induced by an inner product. R.C. James [19] gave the examples

$$\|(x,y)\| = \begin{cases} (|x|^p+|y|^p)^{1/p} & , \text{ if } xy\geq 0 \\ (|x|^q+|y|^q)^{1/q} & , \text{ if } xy<0 \end{cases} \quad (p>0,\ q>0,\ \frac{1}{p}+\frac{1}{q}=1)$$

and finally M.M. Day [11] proved that every possible example is, as the above, a suitable combination, on even and odd quadrants, of an arbitrary norm and its dual (a kind of "**mixed norms**").

Since B-orthogonality is, in general, non symmetric it is necessary to distinguish between **additivity to the right** (the mentioned one) and **additivity to the left** ($x\perp z$, $y\perp z \Rightarrow x+y\perp z$), and also between **existence** and **uniqueness to the right** and **to the left**, in the obvious corresponding sense.

The property of **existence to the right of B-orthogonality** can be viewed as a simple consequence of the following more expressive result [20]:

For every $x\in E$ there exists a closed and homogeneous hyperplane H such that $x\perp^B H$.

This proposition is nothing else than a well known corollary of the Hahn-Banach theorem, taking into account that **a point $x\in E$ is B-orthogonal to other point $y\in E$ if and only if there exists a continuous linear functional $f\in E'\setminus\{0\}$ such that $f(x)=\|f\|\|x\|$, $f(y)=0$** . [20]

It is worthwile to mention in this context the statement in terms of B-orthogonality of a deep theorem of R.C. James [21] which says:

"Let E be a Banach space. E is reflexive if and only if for every closed and homogeneous hyperplane H there exists a point $x\in E\setminus\{0\}$ such that $x\perp^B H$ ".

From another viewpoint it is also proved in [20] that $x\perp^B \alpha x+y$, **($x\neq 0$), if and only if**

$$\lim_{\lambda\to -0} \lambda^{-1}(\|x+\lambda y\|-\|x\|) \leq -\alpha\|x\| \leq \lim_{\lambda\to +0} \lambda^{-1}(\|x+\lambda y\|-\|x\|)$$

from which it follows that the numbers $\alpha=\alpha(x,y)$, for $x\in S$, are the points of a compact interval with the opposite of each derivatives of the norm as ends.

The **existence to the left of B-orthogonality** follows from the homogeneity of this orthogonality and the convexity of the function $\lambda\in\mathbb{R} \longrightarrow \|\lambda x+y\|$, [20].

Therefore it is also true that $\alpha x+y \perp^B x$, $\beta x+y \perp^B x$ and $\alpha<\gamma<\beta$ imply $\gamma x+y \perp^B x$.

However the existence, for every $x\in E$, of a closed and homogeneous hyperplane H such that $H \perp^B x$ is a characteristic property of inner product spaces of dimension ≥ 3, [19]. Infact it is equivalent to the Blaschke-Kakutani theorem.

Having into account the above results it is no difficult to prove the equivalence between the following propositions [20]:

i) B-orthogonality is additive to the right.

ii) B-orthogonality is unique to the right.

iii) For every $x\neq 0$, the closed and homogeneous hyperplane H such that $x \perp^B H$ is unique.

iv) E is smooth. (i.e. the norm is Gateaux-differentiable in every point different of zero, or, in other words, there are no "corners" in the plane sections of the unit sphere.)

Analogously the following propositions are equivalent [20]:

i) B-orthogonality is unique to the left.

ii) For every $x\in S$ and every closed and homogeneous hyperplane H such that $x \perp^B H$, the hyperplane $x+H$ only touchs S in the point x.

iii) E is rotund. (i.e. the norm is strictly convex, or, in other words, there is no segments in the unit sphere.)

For additivity to the left we have that [20]:

If $\dim E=2$, then B-orthogonality is additive to the left if and only if it is unique to the left (E is rotund).

If $\dim E\geq 3$, then B-orthogonality is additive to the left if and only if it is symmetric (E is an inner product space).

Finally B-orthogonality admits unique diagonals, being the corresponding δ such that $3^{-1}\|x\| \leq \delta\|y\| \leq 3\|x\|$. Furthermore E is an inner product space if and only if $\|x\|=\delta\|y\|$ for every $x,y\in E\setminus\{0\}$. [6]

MAIN PROPERTIES OF I-ORTHOGONALITY

It is obvious that **I-orthogonality is symmetric.**

On the other hand **I-orthogonality is either homogeneous or additive if and only if the norm is induced by an inner product.** [18]

Such proposition follows from the Fiken's characterization of inner product spaces already mentioned with regard to the existence of R-orthogonality.

Elementary convexity arguments prove the **existence of I-orthogonality.**

Since I-orthogonality is non homogeneous we cannot say in this case, as for B-orthogonality, that the existence property, just as we have state it, is equivalent to the weaker James's result [18] that for every $x, y \in E$ there exists a number α such that $x \perp\!\!\!\perp \alpha x + y$. However, also as for B-orthogonality, if $x \neq 0$ then the set $\{\alpha: x \perp\!\!\!\perp \alpha x+y\}$ is a non empty compact interval.

Then for non homogeneous orthogonalities we must distinguish between **uniqueness** (in the initial sense of this paper), **α-uniqueness** (the above interval is reduced to a point) and **S-uniqueness** (for every oriented plane P and every $x \in S \cap P$ there exists a unique $y \in S \cap P$ such that the pair $[x,y]$ is in the given orientation and $x \perp\!\!\!\perp y$).

The known results about this are the following:

I-orthogonality is either unique or α-unique [24] **if and only if E is rotund.**

I-orthogonality is S-unique [2].

Finally it is obvious that **I-orthogonality admits unique diagonals.**

MAIN PROPERTIES OF P-ORTHOGONALITY

It is obvious that **P-orthogonality is symmetric.**

It is either homogeneous or additive if and only if the norm is induced by an inner product [18].

In fact it is no difficult to see that additivity implies homogeneity and that this implies the fulfilment of parallelogram law.

As for I-orthogonality elementary convexity arguments prove the **existence of P-orthogonality** [18].

With regard to uniqueness we shall distinguish between the already mentioned three types or degrees for such property:

P-orthogonality is unique if and only if E is rotund.

P-orthogonality is α-unique [24].

We do not know a characteristic property for S-uniqueness. Obviously rotundity of E is a sufficient condition for it, but if E is the space $\mathbb{R}^2$ with a regular octogon as unit sphere then P-orthogonality is S-unique, whereas P-orthogonality is non S-unique when S is an square.

P-orthogonality admits unique diagonals and, in an analogous way to that of B-orthogonality, we have that **the number δ of this property satisfies $2^{-1/2}\|x\| \leq \delta\|y\| \leq (2^{1/2}-1)^{-1}\|x\|$.** Furthermore **E is an inner product space if and only if $\|x\|=\delta\|y\|$ for every $x,y\in E\setminus\{0\}$.** [5]

MAIN PROPERTIES OF S-ORTHOGONALITY

S-orthogonality can be viewed as a normalized I-orthogonality. Then we have that

S-orthogonality is homogeneous, symmetric, existing, unique (I-orthogonality is S-unique) **and with existing unique diagonals.**

As a consequence of the above, **S-orthogonality is additive on two-dimensional spaces.**

We do not know when S-orthogonality is additive in spaces of dimension ≥ 3, but **we conjecture that if $\dim E \geq 3$ and if S-orthogonality is additive then E is an inner product space.**

MAIN PROPERTIES OF C-ORTHOGONALITY

C-orthogonality is symmetric in some cases (Isosceles, Pythagorean, ...) **and it is non symmetric in other cases** (e.g. $x \perp^{C} y$ when $\|x+2y\|=\|x-2y\|$).

We conjecture that either C-orthogonality is trivially

symmetric (as in Isosceles, Pythagorean, ...) **or such property is characteristic of inner product spaces** (as in the above mentioned example).

C-orthogonality is either homogeneous or additive to the left (to the right) **if and only if E is an inner pruduct space** [10].

The very involved proof given in [10] of this proposition is based on the fact that either homogeneity or additivity to the left imply the following characteristic property of inner product spaces

$$x \perp_{C} y \;\Rightarrow\; \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i \|\beta_i nx+\gamma_i y\|^2=0$$

(with $\|\beta_i x+\gamma_i ny\|$ in place of $\|\beta_i nx+\gamma_i y\|$ for additivity to the right).

It is easy to see that **C-orthogonality is existing** (in any sense) **to the right and to the left** [10][2].

With regard to **uniqueness** (in any sense) we only know partial answers as the already mentioned for I and P-orthogonalities and the fact, easy to prove, that **C-orthogonality is non-unique (general sense) when the space is non rotund.**

Moreover **we conjecture that rotundity of the space is not only necessary but a sufficient condition for uniqueness (general sense) of C-orthogonality.**

Finally **C-orthogonality admits diagonals** [13] but we do not know if, in general, they are unique.

MAIN PROPERTIES OF D-ORTHOGONALITY

There is no doubt that **D-orthogonality is homogeneous and symmetric.**

It also satisfies the properties of existence [15] **and existence of diagonals.**

Both properties follow from the analogous properties of B-orthogonality through elementary continuity arguments and the fact that [15]:

"If $x \perp_{B} y$ then $\sup\,(f(x)g(y)-f(y)g(x) : f,g\in S^*) \geq \|x\|\,\|y\|$"

It is proved in [28] that B and D-orthogonality are equivalent when D-orthogonality is either additive or unique and, as a consequence of it, that **in spaces of dimension ≥ 3 the following propositions are equivalent:**

D-orthogonality is additive.

D-orthogonality is unique.

B-orthogonality is symmetric (E is an inner product space and hence B and D-orthogonalities are equivalent).

But also it is true in spaces of dimension 2 that B-orthogonality and D-orthogonality are equivalent if and only if B-orthogonality is simmetric. This fact follows easily from the nature of the Day's "mixed norms" [11], already mentioned in this paper with regard to symmetry of B-orthogonality.

Therefore **in spaces of dimension 2 D-orthogonality is unique if and only if the space is endowed with a rotund and smooth "mixed norm".**

Moreover since it is homogeneous **D-orthogonality is additive in spaces of dimension 2 if and only if it is unique.**

Finally a simple analysis of the convex function

$$\rho\in\mathbb{R}_+ \longrightarrow \rho^{-1}\|x+\rho y\|\,\|x-\rho y\|$$

shows that **diagonals are unique for D-orthogonality if and only if B and D-orthogonality are equivalent,** i.e. "mixed norms" in the 2-dimensional case and i.p.s. for dimension ≥ 3.

MAIN PROPERTIES OF A-ORTHOGONALITY

Firstly we describe a suitable analytical setting for A-orthogonality.

Let S be the unit sphere of a plane (identified to $\mathbb{R}^2$) of E and let $s(\lambda)$ be the point of S that is to a given point $s(0)\in S$ at an angle λ measured with a given orientation.

Then the mapping $s: [0,2\pi] \longrightarrow S$ is a parametrization of the curve S, continuous and of bounded variation, which serves to say that a point $x=\|x\|s(\alpha)$ is A-orthogonal to other point $y=\|y\|s(\beta)$, linearly independent with it, if and only if

$$\int_{\alpha}^{\beta} s_1(\lambda)ds_2(\lambda)-s_2(\lambda)ds_1(\lambda) = \int_{\beta}^{\alpha+\pi} s_1(\lambda)ds_2(\lambda)-s_2(\lambda)ds_1(\lambda)$$

(obviously there is no essential restriction in supposing $0\leq\alpha<\beta\leq\pi$)

It follows from the above analytical setting (also from the ingenuous geometrical thought) that **A-orthogonality is homogeneous, symmetric, existing, unique and with existing unique diagonals.**

As in the case of S-orthogonality it follows from the homogeneity and uniqueness that **A-orthogonality is additive in spaces of dimension 2.**

Also as for S-orthogonality **we conjecture that additivity of A-orthogonality is a characteristic property of inner product spaces of dimension ≥ 3.**

MAIN PROPERTIES OF ORTHOGONALITIES IN NORMED LINEAR SPACES

	ROBERTS	BIRKHOFF	ISOSCELES	PYTAGOREAN	SINGER	CARLSSON	DIMINNIE	AREA
NONDEGENERACY SIMPLIFICATION CONTINUITY	always	always	always	always	always	always	always	always
HOMOGENEITY	always	always	i.p.s.	i.p.s.	always	i.p.s.	always	always
SYMMETRY	always	dim E=2: "mixed norms" dim E≥3: i.p.s.	always	always	always	i.p.s. ? except obvious cases	always	always
ADDITIVITY non-symmetry: left/right	trivial cases	left: dim E=2: rotund; dim E≥3: i.p.s. right: smooth	i.p.s.	i.p.s.	dim E=2: always dim E≥3: i.p.s. ?	i.p.s.	dim E=2: rotund smooth "mixed norms" dim E≥3: i.p.s.	dim E=2: always dim E≥3: i.p.s. ?
EXISTENCE	i.p.s.	always	always	always	always	always	always	always
UNIQUENESS non-symmetry: left/right non-homogeneity: uniqueness α-uniqueness S-uniqueness	always	left: rotund right: smooth	rotund α:rotund S:always	rotund α:always S: ?	always	rotund ? α: ? S: ?	dim E=2: rotund smooth "mixed norms" dim E≥3: i.p.s.	always
EXISTENCE DIAG.	i.p.s.	always	always	always	always	always	always	always
UNIQUENESS DIAGONALS	always	always	always	always	always	?	dim E=2: "mixed norms" dim E≥3: i.p.s.	always

(X ? means: "we conjecture that X").(? means: "open question without conjecture")

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