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SOME TOPOLOGICAL PROPERTIES OF STABLE NORMS

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A p-Banach space $(X,\|.\|)$ $0<p\leq 1$, is said to be stable if

$$(*)\quad \lim_{n,U}\lim_{m,V}\| x_n+y_m\| = \lim_{m,V}\lim_{n,U}\| x_n+y_m\|$$

whenever $(x_n)_n$ and $(y_m)_m$ belong to $B=\{x\in X;\ \| x \| \leq 1\}$ and U, V are non-trivial ultrafilters on $\mathbb{N}$. If $(*)$ holds for every weakly compact subset B of X, then $(X,\|.\|)$ is called a weakly stable p-Banach space. The stable Banach spaces were introduced by Krivine and Maurey in [8], and their main property is that they contain l_p for some p, $1\leq p<\infty$,. The stable p-Banach spaces were considered for the first time in [2] and Krivine-Maurey's result was extended to this setting.

The weakly stable Banach spaces have been recently defined in [1] and they contain l_p or c_o. Most of the usual Banach spaces are stable (some of them appear in [6]); c_o, the typical non-stable Banach space, is weakly stable. If $(x_n)_n$ is a sequence on c_o w-convergent to y, then, by passing to a suitable subsequence, $x_n = y_n + z_n$ where supp $y_n \cap$ supp $z_n = \emptyset$, $(y_n)_n$ converges to y and $(z_n)_n$ is a block basis sequence, [4]. This enables a representation of the types on c_o defined by weakly convergent sequences as $\tau(x) = \lim_{n,U}\| x+x_n\|_o = \max(\| x+y \|_o,\ a)$ with $a = \lim_{n,U}\| z_n\|_o$; wich gives the weakly stability of $(c_o,\|.\|_o)$; (in [1] a different proof is given).

Similar techniques can be used to prove the weakly stability of the Orlicz spaces h_M, and the modular spaces, when M is a non degenerate p-convex Orlicz function wich does not verify the Δ_2 condition.

The stability (weakly stability) depends on the norm considered on X. In fact, for each stable space equivalent non-stable "norms" may be built. This was done for l_p in [7], for stable Banach spaces with unconditional basis in [11] and for L_1 in [3]. In this note we report the general result [5] and some related things.

Some notations are needed. As usualy, the Banach-Mazur distance between isomorphic p-Banach spaces is defined by $d(X,Y)=\inf\{\| T \| \| T^{-1}\| ; T: X \longrightarrow Y$ is an onto isomorphism$\}$. Note that $d(X,Y)=1$ does not imply that X and Y are isometric. We denote by N the associated quotient metric space consisting of equivalent classes, modulo distance equal to one, of all the norms on X wich are equivalent to the original one.

If X is an infinite dimensional stable (weakly stable) p-Banach space, the set $S=\{\| . \| \in N ; \| . \|$ is stable$\}$ is closed and nowhere dense in N.

Observe that there is not ambiguity in the above proposition because of all the "norms" belonging to a class are either stable or non-stable at the same time.

If $\|.\|_1$ and $\|.\|_2$ belong to S and $d(\|.\|_1, \|.\|_2) > 1$ then $\phi(t) = ((1-t) \|.\|_1^p + t \|.\|_2^p)^{1/p}$ is a p-convex continuous mapping from [0,1] into S connecting $\|.\|_1$ and $\|.\|_2$, so, **S is arcwise conected.** Thus, **the stable "norms" are limit points of stable "norms" whenever # N > 1.** We think that the last condition is redundant and the diameter of S is always infinite, but, at present, we have only been able to prove it for stable spaces having a non-trivial cotype.

However the existence of a non-trivial cotype is not necesary in order to ensure $\# \mathbf{N} > 1$. For instance, when $X = l^1(l^\infty_n)$ it is shown in [5] that $\# \mathbf{N} > 1$.

The same property holds for the weakly stable space c_o. Using similar ideas to those in our proof of weakly stability of c_o, it can be shown that $\|x\| = \|x\|_o + |e^*_1(x)| + \max\{|(e^*_k - e^*_{k+1})x| \; ; \; k>1\}$ is an equivalent weakly stable norm on c_o. If $\|x\| \leq \|Tx\|_o \leq C\|x\|$, by considering the type defined by $(T(e_n - e_{n-1}))_n$ on $(c_o, \|.\|_o)$ and the representation of the types in this space, it is easy to see that $C > 4/3$.

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