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Approximation Properties Defined by Operator Ideals

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Let E be a locally convex space (lcs) and **A** an ideal of operators between lcs; we are mainly interested in **A**=**F** (finite rank operators) and **A**=**K** (compact operators).

Definition. E is said to possess the Uniform Approximation Property with respect to **A**, (in short **A**-UAP), when for each $U \in U(E)$ there exist a $V \in U(E)$ and a sequence $(T_n) \subset \mathbf{A}(E,E)$ such that

$$p_U(x - T_n x) \leqslant n^{-1} p_V(x) \qquad x \in E$$

and it is said to have the co-**A**-UAP when for each $A \in B(E)$ there exist a $B \in B(E)$ and a sequence $(T_n) \subset \mathbf{A}(E,E)$ such that

$$p_B(x - T_n x) \leqslant n^{-1} \qquad x \in A$$

Recalling that **G** denotes the ideal of operators acting between Banach spaces which are approximable (in the operator norm) by finite rank operators, we have proved in |2| that E has the **F**-UAP iff E is a **G**-space and that E has the co-**F**-UAP iff E'_b is a **G**-space. Combining this with |7|, we establish the relation between the **F**-UAP and the classical Approximation Property (AP):

Theorem 1. |2|. Let E be an FM space; then are equivalent:
1. E has AP 2. E has co-F-UAP 3. E'_b has F-UAP 4. E'_b has AP

when we pass to the choice **A**=**K** we have

Theorem 2. |4| Let E be an FM space. Then are equivalent:
1. E has CAP 2. E has co-K-UAP 3.E'_b has K*-UAP 4. E'_b has CAP

where CAP denotes the Compact Approximation Property of |9|. It is worth to remark that in this case not only E'_b has **K**-UAP, but the sequence of compact operators is "collectively compact" -see |5|-.

We turn again to the **F**-UAP, which we sill simply call UAP. In |1| Benndorf proves: **Let E be a Fréchet Schwartz space with the Bounded Approximation Property (BAP), and λ a sequence space. Then there exists a partition of the identity in E absolutely- λ-summable. If E possesses an FDD, then E also possesses an absolutely- λ-summable FDD.**

His proof is based upon the result of Pelczynski and Wojtaszczyk: "Every Fréchet space with the BAP is isomorphic to a complemented subspace of a Fréchet space with an FDD".(|8|)

In |3| we showed that the core of the proof is actually the UAP, by giving a purely internal proof of Benndorf's result without appeal to Pelczynski's theorem; the idea is that once the UAP is obtained, by "diagonalization" of the sequences of operators we may reach any prefixed order of convergence with respect to 0-nbhd.

Analogously, we will have that in a lcs with a fundamental system of bounded sets and the co-UAP, again by "diagonalization" we could reach any prefixed order of convergence with respect to bounded sets; recalling that in |6| is defined for a sequence (x_n) in a lcs E to be boundedly- λ -summable when there exists a bounded set B in E such that $(p_B(x_n)) \in \lambda$, we may prove:

Theorem 3. |3|. Let E be a complete DF co-Schwartz space with BAP, λ a sequence space. Then E possesses a boundedly- λ-summable partition of the identity. If E possesses an FDD, then E also possesses a boundedly- λ-summable FDD.

dualizing in this way Benndorf's result. It is clear that in both cases the difficulties arise in proving that BAP implies UAP and co-UAP (compare this with Theorem 1).

Replacing again the ideal **F** by the ideal **K** of compact operators, we have the notions of **K**-BAP (a sequence of compact operators pointwise convergent to the identity), compact partition of the identity and **K**-Decomposition, and may prove:

Theorem 4 |4|. Let E be a Fréchet Schwartz space (resp. a complete DF co-Schwartz space), and λ a sequence space. Then E

possesses the K-BAP then E also has an absolutely- λ -summable compact partition of the identity (resp. a boundedly- λ -summable compact partition of the identity). If E possesses a K-D, then E also possesses an absolutely- λ-summable (resp. boundedly- λ -summable) KD.

Compare again this with theorem 2.

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(This paper contains the main results of |3| and |4|)