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THE LAST COEFFICIENT OF THE SAMUEL POLYNOMIAL

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Let X be an r-dimensional noetherian scheme, proper over an Artinian ring A. Let Y be a closed subscheme of X defined by a sheaf of ideals I, and let $\pi:\overline{X}\longrightarrow X$ be the blowing-up of X with respect to I.

For sufficiently large n, Ramanujam[6] proved that

$$S_I(n) = \chi(X,\mathcal{O}_X/I^n) = \sum_{i=o}^{r}(-1)^i \mathrm{length}_A H^i(X,\mathcal{O}_X/I^n)$$

is a polynomial in n of degree $\leq r$. Moreover, if E is the exceptional divisor of π, the leading coefficient of this polynomial is

$$\mathrm{degree}(I\mathcal{O}_{\overline{X}}\otimes_{\mathcal{O}_{\overline{X}}}\mathcal{O}_E)/r! = -\mathrm{degree}(I\mathcal{O}_{\overline{X}})/r!$$

Every coefficient of $S_I(n)$, except the last one, can be computed in terms of the exceptional divisor E. In this paper we proved that the last coefficient of $S_I(n)$ is the difference in the arithmetic genus: $\chi(X,\mathcal{O}_X)-\chi(\overline{X},\mathcal{O}_{\overline{X}})$. By the way we obtain another proof of Ramanujam's result and a calculation of all the coefficients. The precise result is

Theorem : *For sufficiently large n*

$$S_I(n) = \chi(X,\mathcal{O}_X) - \chi(\overline{X},\mathcal{O}_{\overline{X}}) - \sum_{i=1}^{r}(-1)^i\chi(E^i)\binom{n}{i}$$

The intersections E^i *are to be taken in the Grothendieck ring* $K^{\cdot}(\overline{X})$ *of coherent locally free sheaves on* $\overline{X}$.

We give the following two consequences

Theorem : Let H be an hypersurface of a smooth ambient variety Z, proper over k. Let $\pi:\overline{H}\longrightarrow H$ be the blowing-up with center a closed point x of H If d is the dimension of Z at x, and m is the multiplicity of H at x, then

$$\chi(H,\mathcal{O}_H) - \chi(\overline{H},\mathcal{O}_{\overline{H}}) = (-1)^{d-1}\binom{m}{d}$$

Theorem : Let $\pi:\overline{X} \longrightarrow X$ be the blowing-up with respect to I, if the Hilbert function $H_I(n) = S_I(n+1) - S_I(n)$ is a polynomial for all $n \gg 0$, then

$$\chi(X,\mathcal{O}_X) = \chi(\overline{X},\mathcal{O}_{\overline{X}})$$

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