EXTENDING CONTRACTIONS AND ISOMETRIES

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In [1] , Aronszajn and Panitchpakdi characterized the **hyperconvex** metric spaces, that is, those spaces X such that given a metric space Z, and a subspace $Y \subset Z$, any contractive mapping $Y \to X$ can be extended to a contraction $Z \to X$, as follows: X is hyperconvex if and only if for any collection of closed balls in X,
$\{B(x_i, r_i) : i \in I\}$, with

$$d(x_i, x_j) \leq r_i + r_j \qquad i, j \in I,$$

the intersection $\bigcap \{B(x_i, r_i) : i \in I\}$ is not empty.

This property is closely related to Nachbin's **binary intersection property**; in fact a metric space X is hyperconvex if and only if it has the binary intersection property and is **metrically convex**, i.e., for any $x, y \in X$ and any $\lambda \in (0,1)$, there exists some z in X with

$$d(z,x) = \lambda \, d(x,y) \quad \text{and} \quad d(z,y) = (1-\lambda) \, d(x,y).$$

Given the renewed interest in the concept of ultrametricity as documented by [4] , we have solved the analogous problem among ultrametric spaces.

Since there are no hyperconvex ultrametric spaces (no nontrivial ultrametric space is metrically convex), we have studied the same property within the category of ultrametric spaces.

THEOREM 1.- **For an ultrametric space X, the following properties are equivalent:**
(i) **Given any ultrametric space Z, and a subspace Y of Z, any contraction $Y \to X$ can be extended to a contraction $Z \to X$.**
(ii) **X has the binary intersection property.**

In the literature, ultrametric spaces with the binary

intersection property are called **spherically complete** spaces.

Recently, [3] , Dress has studied the related problem of extending isometries between metric spaces.

Calling an extension Y of a metric space **tight** when for all y_1, y_2 in Y,

$$d(y_1, y_2) = \sup \{d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2): x_1, x_2 \in X\},$$

and a metric space **fully spread** when it has no proper tight extensions, Dress has proved, among other things, the following: if X is fully spread, and Z is a tight extension of Y, then any isometric embedding $Y \to X$ can be extended to an isometric embedding $Z \to X$ (see [3] , th. 3 (vi) and (vii)). Then,

THEOREM 2.- (a) **A metric space is fully spread if and only if it is hyperconvex.**

(b) **If $Z \supset Y$ is not a tight extension of Y, then there is a fully spread space X and an isometric embedding $Y \to X$ that cannot be extended to an isometric embedding $Z \to X$.**

And coming back to the ultrametric setting, we say that an ultrametric space Y is an **ultrametrically tight** extension of $X \subset Y$ when

$$\forall\ y_1, y_2 \in Y,\quad d(y_1, y_2) > \inf \{d(y_1, x): x \in X\}.$$

Then,

THEOREM 3.- **For any ultrametric space X, the following properties are equivalent:**

(i) If Y is an ultrametric space and $Z \supset Y$ is an ultrametrically tight extension of Y, then any isometric embedding $Y \to X$ can be extended to an isometric embedding $Z \to X$.

(ii) X is spherically complete.

A result analogous to Theorem 2 (b) is also valid in the ultrametric case.

The proofs of Theorem 1 and 3 will appear in [2] .

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