EXTRACTA MATHEMATICAE 1, n.2, 96-98 (1986)

SOME CHARACTERISTIC AND NON-CHARACTERISTIC PROPERTIES OF INNER PRODUCT SPACES

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AMS Class.: 46B20, 46C10

INTRODUCTION

Let E be a real or complex normed linear space with unit sphere $S=\{x\in E : \|x\|=1\}$ and let $\lambda>0$, $0<\varepsilon<2$. We say that E satisfies, respectively, the properties P_λ, Q_ε, R_ε if

$$P_\lambda : \quad x,y\in S, \quad \|x+\lambda y\|=\|x-\lambda y\| \quad \Longrightarrow \quad \|x+\lambda y\|^2=1+\lambda^2$$

$$Q_\varepsilon : \quad x,y\in S, \quad \|x-y\|=\varepsilon \quad \Longrightarrow \quad \|x+y\|^2=4-\varepsilon^2$$

$$R_\varepsilon : \quad \delta(\varepsilon)=1-(1-\varepsilon^2/4)^{1/2}$$

where $\delta(\varepsilon)=\inf\{1-\|x+y\|/2 \ : \ x,y\in S, \ \|x-y\|=\varepsilon\}$ denotes the modulus of convexity of E.

It is well known that inner product spaces satisfy the above properties for every λ and ε. On the other hand Borwein and Keener [4] and Nordlander[10] conjecture, respectively, that either the fulfilment of P_λ or R_ε for any λ or ε is a characteristic property of inner product spaces.

For $\lambda=\varepsilon(4-\varepsilon^2)^{-1/2}$ the above properties are equivalent and we prove that the mentioned conjectures are true for almost every λ and ε, but they are false (at least when E is real and two dimensional) for λ and ε belonging to a countable and dense subset of $\mathbb{R}_+$ and (0,2) respectively.

In particular we prove that the conjecture is true for the case $\lambda=2$ specially considered in [4] in connection with some problems relative to Chebyshev centers. With this and the paper of Amir and Mach [2] all the conjectures and open questions posed in [4] and [10] are solved.

RESULTS

PROPOSITION 1. The properties P_λ, Q_ε, R_ε are equivalent when $\lambda=\varepsilon(4-\varepsilon^2)^{-1/2}$.

PROPOSITION 2. If E satisfies the property P_λ for any $\lambda>0$ such that

$$\lambda\notin D=\{\tan(k\pi/2n) \ : \ n=1,2,\dots; \ k=1,2,\dots,n-1\}$$

then E is an inner product space.

REMARK. If the unit sphere of a norm in $\mathbb{R}^2$ is invariant under rotations of angle $\pi/2n$, $(n=2,3,\dots)$, and if $\lambda=\tan(k\pi/2n)$, $(k=1,2,\dots,n-1)$, then such normed linear space satisfies the propertiy P_λ. Thus for every $\lambda\in D$ there exist real 2-dimensional non inner product spaces satisfying the property P_λ. This is the case of the linear space $\mathbb{R}^2$ endowed with the norm whose unit sphere is the regular 4n-gon.

CONJECTURE. For every $\lambda>0$ the property P_λ characterizes the real inner product spaces of dimension $\geqslant 3$.

PROPOSITION 3. $2\notin D$.

REMARK. Following S. O. Carlsson [5] and R. C. James [7] we can say that x is λI-orthogonal to y, $x \perp_{\lambda I} y$, when $\|x+\lambda y\|=\|x-\lambda y\|$ and that x is λP-orthogonal to y, $x \perp_{\lambda P} y$, when $\|x+\lambda y\|^2=1+\lambda^2$.

Then we can paraphrase the Proposition 2 by saying that for every $\lambda\notin D$ the property

$$x,y\in S, \quad x \perp_{\lambda I} y \quad \Rightarrow \quad x \perp_{\lambda P} y$$

is characteristic of the inner product spaces.

With this formulation the Proposition 2 is in the line of many results of characterization of inner product spaces based in the relation between various tipes of generalized orthogonality in normed linear spaces [1], [6], [8], [9], [11].

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(To appear in Journal of Approximation Theory.)