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BANACH ALGEBRAS WHICH ARE A DIRECT SUM OF DIVISION ALGEBRAS

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Recently Esterle and Oudadess [1] have shown that a complex semisimple Banach algebra A satisfying $Ax^2 = Ax$ for every $x \in A$ is isomorphic to $\mathbb{C}^n$ for some $n \geq 0$. Their strategy consists of proving that such an algebra is commutative and then to obtain the result by using standard techniques of spectral theory. We note that these methods are not suitable in order to extend the result to the case of a real Banach algebra, since there exist noncommutative real Banach algebras (for instance, the quaternions) which are semisimple and satisfy $Ax^2 = Ax$.

By using techniques of the theory of von Neumann regular algebras [3] we have proved the following result:

THEOREM [2]. *Every (real or complex) semiprime Banach algebra A satisfying $xAx = x^2Ax^2$ for every $x \in A$ is a direct sum $A = M_1 \oplus .. \oplus M_n$ of ideals each of which is isomorphic to either the reals, complexes or quaternions. If A is complex all the M_i are isomorphic to the complex field and A is commutative. Conversely, every direct sum $A = M_1 \oplus ... \oplus M_n$ of division Banach algebras is a semisimple Banach algebra satisfying $xAx = x^2Ax^2$ for every $x \in A$.*

REMARK. Since a complex semisimple Banach algebra A satisfying $Ax = Ax^2$ for every $x \in A$ is commutative [1, Lemma 3.1], then $xAx = x^2Ax^2$, as it is not difficult to see, so that we can apply our theorem to get $A \cong \mathbb{C}^n$ for some $n \geq 0$, giving so a new proof of the result of Esterle and Oudadess.

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