EXTRACTA MATHEMATICAE 1, 47-49 (1986)

C-COMPACT CONVEX SETS AND EXTREME POINTS

J. Martinez-Maurica and C. Pérez García
Facultad de Ciencias. Avda. Los Castros. Santander, Spain.

A classical open problem in nonarchimedean analysis has been the development of an ultrametric Krein-Milman theory (see [3]). This problem is solved in [2] for compact convex sets by giving the following definition of extreme points,

DEFINITION 1. Let E be a vector space over a valued field K (archimedean or not) and A a subset of E. A nonempty part S of A is said to be an extreme set of **A** if : i) S is semiconvex (i.e. $\lambda S+(1-\lambda)S \subset S$ for all $\lambda \in K$ with $|\lambda| < 1$),and ii) If the convex hull of a finite set $\{x_1,..,x_n\} \subset A$ has nonempty intersection with S, then there is $i \in \{1,..,n\}$ such that $x_i \in S$. A point $x \in A$ is called an extreme point of A if it belongs to some minimal extreme set of A.

Let A be a compact convex set with more than one point of a separated locally convex space E and let S be a minimal extreme set of A. Then S is reduced to be a single point if and only if K is archimedean (theorems 1 and 4 of [2]).

If K = R or C, definition 1 gives the same extreme points as the usual ones (according to the definition of Kalton [1]) for every p-convex compact set of a separated locally p-convex space (theorem 1 of [2]) ($p \in (0,1]$).

If K is nonarchimedean, definition 1 allows us to obtain the following theorem of Krein-Milman,

THEOREM 2 ([2] theorem 3). Every compact convex set of a separated locally convex space over K is the closed convex hull of its extreme points.

Nevertheless, compactness is quite a restrictive condition in nonarchimedean analysis since it requires the ground field to be locally compact. Hence we propose in [4] a new definition of extreme points for the more general class of c-compact convex sets which were introduced by Springer in 1965

DEFINITION 3. Let E be a topological vector space over K and A a 0-convex set in E (i.e. A is a convex set with $0 \in A$). A nonempty part S of A is said to be a 0-extreme set of A if there exists $K_1 \subset K$ such that $S = \bigcap_{\alpha \in K_1} \{x \in A: f_\alpha(x)= \alpha\}$ with

$f_\alpha \in E'$ satisfying $|f_\alpha(A)| \leq |\alpha|$, for all $\alpha \in K_1$. A point $x \in A$ is called a 0-extreme point of A if it belongs to some minimal element of $F = \{S \subset A$: S is a proper 0-extreme set of A $\}$. The set of 0-extreme points of A is denoted by $Ext^o(A)$.

Let E be a separated locally convex space over K and let A be a 0-convex m*-closed (i.e. for every $f \in E'$, f(A) is K or $\{x \in K: |x| \leq |\lambda|\}$ for certain $\lambda \in K$), c-compact and bounded set of E. Then, for one such A we have the following Krein-Milman theorem, ([4], theorem 6),

THEOREM 4. A is the closed 0-convex hull of its 0-extreme points.

If E is a nonarchimedean normed space over K, then there exists an orthogonal sequence $(e_n)_{n \in N}$ in E with $\lim \|e_n\|=0$ such that $A = \{\sum x_n e_n : |x_n| \leq 1\}$ for all n . In this case we have,

THEOREM 5 ([4] theorem 7). For a point $b = \sum b_n e_n$ of A, the following properties are equivalent,

(a) $\{b\}$ is a 0-extreme set of A.

(b) b is a 0-extreme point of A.

(c) There exists $f \in [A]' - \{0\}$ such that $|f(b)| = \sup_{x \in A} |f(x)|$ (where [A] stands for the linear hull of A).

(d) There exists $n \in N$ such that $|b_n|=1$.

A similar theorem works for locally convex spaces (see [4], theorem 9).

Theorem 5 implies that every minimal 0-extreme set of A is reduced to be a single point. As another consequence of theorem 5 we can also prove that for every compact 0-convex set A of a separated locally convex space E over an ultrametric local field K, $Ext^o(A)$ agrees with the set of extreme points of A according to definition 1 (see [4], theorem 11).

It is natural to ask what the definition of extreme point is for a convex set A in a topological vector space E. Let A_o be the 0-convex set such that $A = x + A_o$ for each x of A. Since it is not true in general that $x + Ext^o(A) = y + Ext^o(A)$ for $x, y \in A$, we must consider a fixing point $x \in A$ and give the following definition: If A is a x-convex set in E, the set of x-extreme points of A is defined by $x + Ext^o(A_o)$. Then, every result in this paper has an analogous for x-convex sets.

Our theorem 5 is quite similar to theorem 2 of Kalton's paper [1] , which establishes that every point of a compact p-convex set $(0 < p < 1)$ subset A of a separated topological vector space E can be expressed in the way $x = \Sigma a_n x_n$ with $a_n \geq 0$, $\Sigma a_n^p = 1$ and (x_n) being a sequence of distinct p-extreme points of A.

REFERENCES

1. KALTON, N.J.: Compact p-convex sets. Quart. J. Math. Oxford Ser. (2) . 28. no.111 (1977),301-308.

2. MARTINEZ-MAURICA,J.; PEREZ GARCIA,C.: A new approach to the Krein-Milman theorem. Pacific J. Math.,120,no.1 (1985).

3. MONNA,A.F.: Rapport sur la théorie des espaces linéaires topologiques sur un corps valué non-archimedien. Bull. Soc. Math. France. Mémoire 39-40,(1974), 255-278.

4. PEREZ GARCIA, C.: The Krein-Milman theorem in non-archimedean analysis. Houston J. of Math. (to appear).

1980 AMS SUBJECT CLASSIFICATION: 46P05