EXTRACTA MATHEMATICAE 1, 35-37 (1986)

A LAGRANGIAN APPROACH TO OPTIMAL STOPPING

Gerardo Sanz Sáiz
Departamento de Estadística Económica
Facultad de Ciencias Económicas y Empresariales
Universidad de Zaragoza

Let $(Z_t, \mathcal{F}_t;\ t \geqslant 0)$ be a "cadlag" gain process such that $|Z_S|$ is integrable for every stopping time (s.t.) S. Let consider a subclass $\mathbb{L}'$ of s.t. included in another class $\mathbb{L}$ which, sometimes, may be the class of all s.t. .

Let denote by $\mathcal{X}$ a family collection of random variables named perturbations of (Z_t), i.e. :

$$\mathcal{X} = (X = (X_T)\ ;\ X_T \in \mathcal{F}_T\ ,\ T \in \mathbb{L})\ .$$

Assume that:

(X1) : For all families (X_T) : $E(X_T^-) < \infty$, $T \in \mathbb{L}$.

(X2) : There exists a family $(X_T°)$ such that:

$$E(X_T°) = 0\ ,\quad T \in \mathbb{L}\ .$$

(X3) : For all families (X_T) : $X_T \geqslant 0$ a.e., if $T \in \mathbb{L}'$.

(X4) : For every s.t. $T \in \mathbb{L} - \mathbb{L}'$ there exists a sequence $(X_T^k\ ,\ k \in \mathbb{N})$ of elements of families from $\mathcal{X}$ such that:

$$E(X_T^k) \xrightarrow[(k \uparrow \infty)]{} -\infty\ .$$

For every $T \in \mathbb{L}$ and $X \in \mathcal{X}$, define the lagrangian function by:

$$L(T, X) = E(Z_T) + E(X_T)\ .$$

Let be:

$$L_*(T) = \inf_{X \in \mathcal{X}} L(T, X)\ ,\quad T \in \mathbb{L}\ ,$$

$$L^*(X) = \sup_{T \in \mathbb{L}} L(T, X)\ ,\quad X \in \mathcal{X}\ ,$$

AMS 1980 subject classification: 60G40, 62L15, 49A60 .

$$L_* = \sup_{T \in \mathbb{L}} L_*(T) \quad , \qquad L^* = \inf_{X \in \mathcal{X}} L^*(X) \quad .$$

To solve the <u>primal problem</u> and/or the <u>dual problem</u> is to find $T^* \in \mathbb{L}$ and/or $X^* \in \mathcal{X}$, if it exists, such that:

$$L_* = L_*(T^*) \quad \text{and/or} \quad L^* = L^*(X^*) \quad .$$

An <u>optimal stopping problem with constraints</u> is to find $T^* \in \mathbb{L}'$, if it exists, for wich:

$$E(Z_{T^*}) = \sup_{T \in \mathbb{L}'} E(Z_T) \quad .$$

Finally, a pair (T^*, X^*), $T^* \in \mathbb{L}$, $X^* \in \mathcal{X}$, is said to satisfy the <u>optimality conditions</u> if :

i) $T^* \in \mathbb{L}'$, $X^* \in \mathcal{X}$,

ii) $L(T^*, X^*) = L^*(X^*)$,

iii) $X^*_{T^*} = 0$ a.e. .

Thus we have:

<u>THEOREM</u> .- The following statements are equivalent :

1.- *The pair* (T^*, X^*) *satisfies the optimality conditions.*

2.- *The pair* (T^*, X^*) *is a saddle-point of the lagrangian function, i.e. :*

$$L(T, X^*) \leqslant L(T^*, X^*) \leqslant L(T^*, X) \quad .$$

3.- *The s.t.* T^* *solves the primal problem, the perturbation* X^* *solves the dual problem and* $L_* = L^*$.

Moreover, in any of these cases, the optimal stopping problem with constraints satisfies :

$$E(Z_{T^*}) = \sup_{T \in \mathbb{L}'} E(Z_T) = \sup_{T \in \mathbb{L}} E(Z_T) = L_* = L^* \quad .$$

#

<u>Remark</u> .- The theorem extends without difficulty to optimal stochastic control problems with constraints. The optimal stopping problem above is no more than a special case of them, with controls $u(t) = 1_{\{U < t\}}$, constructed from stopping times U .

Remark .- The theorem includes optimal stopping problems with constraints such as the following:

1.- $\mathbb{L}' = \{ T \in \mathbb{L} , U \leqslant T \leqslant V$ a.e. $\}$, $U , V \in \mathbb{L}$, $U \leqslant V$ a.e. (Nachman (1980)).

2.- $\mathbb{L}' = \{ T \in \mathbb{L} , E(T) \leqslant \alpha \}$, $\alpha \in R^{+} \cup \{0\}$ (Kennedy (1982)).

3.- $\mathbb{L}' = \{ T \in \mathbb{L} , E(Y_T) \geqslant \sup_{S \in \mathbb{L}} E(Y_S) - a \}$, $a \geqslant 0$, being (Y_S) a process with the same characteristics as (Z_S) and $\sup_{S \in \mathbb{L}} E(Y_S) < \infty$ (Pontier (1983)).

4.- The "stochastic game" of Dynkin (Dynkin (1969)).

References

1.- Dynkin, E.B. (1969): *Game variant of a problem on optimal stopping*. Soviet Math., 10, 270 - 274.

2.- Kennedy, D.P. (1982): *On a constrained optimal stopping problem*. Journal Appl. Prob., 19, 631 - 641.

3.- Nachman, D.C. (1980): *Optimal stopping with a horizon constraint*. Math. Oper. Res., 5, 126 - 134.

4.- Pontier, M. et Szpirglas, J. (1983): *Arrêt optimal avec contrainte*. Adv. Appl. Prob., 15, 798 - 812.