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ON SUPERSOLUBLE GROUPS ACTING ON KLEIN SURFACES

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by

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1. Introduction. It is well known that a group of automorphisms G of a compact Klein surface X of algebraic genus $p \geq 2$ has at most $84(p-1)$ elements **[11]**. Although this bound is known to be attained for infinitely many values of **p** **[5,6,7,13,14,19]** it is no longer true when X is assumed to have nonempty boundary as well as when one put some specific conditions for G to be satisfied. In other words let $\mathscr{F}$ be a family of finite groups and let $\mathscr{K}$ be a class of compact Klein surfaces. Denote by $N(p,\mathscr{F},\mathscr{K})$ the order of a biggest group from $\mathscr{F}$ that a Klein surfaces from $\mathscr{K}$ of algebraic genus $p \geq 2$ admits as a group of automorphisms. The following problems arise naturally in this context.

(1) Find the upper bound (in terms of **p**) for $N(p,\mathscr{F},\mathscr{K})$.

(2) Having $N(p,\mathscr{F},\mathscr{K})$, describe those **p** for which this bound is attained.

(3) Describe the topological types (called *species*) of the corresponding Klein surfaces.

(4) Describe the algebraic structure of the corresponding groups of automorphisms.

The number of results concerning the group of automorphisms of compact Klein surfaces obtained in the recent years can be read off within this schema. Although these results have evidently topologically-analytic flavor they essentially concern certain purely group theoretical problems. The most complete results were obtained for the family of supersoluble groups. Results made a fairly nice picture, some of them are new or are still in print and the aim of this talk is to give a survey of them together with some comments concerning proofs.

2. Methods. By a Klein surface we shall mean a compact topological surface equipped with dianalytic structure. Although this object seems to be already known to Klein **[12]**, its modern study is due to Alling and Greenleaf **[1]**. By an automorphism of a Klein surface we mean a conformal or anticonformal homeomorphism. Every Klein surface X of algebraic genus $p \geq 2$ can be represented as a quotient space D/Γ, where D is the upper complex half plane with hyperbolic structure and Γ is a certain discrete group of isometries of D including those which reverse orientation; reflections and glide-reflections (*NEC group*) **[18]**. Moreover Γ can be assumed to be a *surface NEC group* i.e. an NEC group with signature $(g;\pm;[-];\{(-),\overset{k}{\ldots},(-)\})$ (see below). The number g corresponds to the topological genus, the sign to the orientability and k is just the number of boundary components of X. If X is not a Riemann surface *i.e.* if the sign is "-" or $k>0$ then the number $p=\alpha g+k-2$ (where $\alpha=2$ if the sign is "+" and $\alpha=1$ otherwise) is called the *algebraic genus of* X (or that of Γ) and it is the same as the topological genus of the canonical double Riemann cover of X **[1]**. In case of Riemann surfaces the algebraic genus **p** equals the topological one g.

On the other hand having a surface so represented a finite group G can be faithfully represented as a group of its automorphisms if and only if there exists an NEC group Λ such that $G \cong \Lambda/\Gamma$ **[14,16,20]**.

The algebraic structure of Λ is determined by its signature **[15]**,**[21]**, being a sequence of numbers and symbols of the following form

$$(g;\pm;[m_1,\ldots,m_r];\{(n_{i1},\ldots,n_{is_i})_{i=1,\ldots,k}\}). \tag{2.1}$$

A group Λ with signature (2.1) has the presentation with the following generators

(i) x_i, $i=1,..,r$,

(ii) c_{ij}, $i=1,..,k$, $j=0,..,s_i$,

(iii) e_i, $i=1,..,k$,

(iv) a_i,b_i, $i=1,..,g$ (if the sign is "+")

d_i, $i=1,..,g$, (if the sign is "-")

subject to the relations

(i) $x_i^{m_i}=1$, $i=1,..,r$,

(ii) $c_{is_i}= e_i^{-1}c_{i0}e_i$, $i=1,..,k$,

(iii) $c_{ij-1}^2=c_{ij}^2=(c_{ij-1}c_{ij})^{n_{ij}}=1$, $i=1,..,k;j=1,..,s_i$,

(iv) $x_1 ...x_re_1...e_ka_1b_1a_1^{-1}b_1^{-1}...a_gb_ga_g^{-1}b_g^{-1}=1$ if the sign "+",

$x_1...x_re_1...e_kd_1^2...d_g^2=1$ if the sign is "-".

We shall refer to above generators as to the *canonical generators* of Λ.

Every NEC group has a *fundamental region* whose hyperbolic area depends only on the algebraic structure of a group and for a group with signature (2.1) is given by

$$\mu(\Lambda)=2\pi(\alpha g+k-2+\sum_{i=1}^{r}(1-1/m_i)+\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1-1/n_{ij})/2). \qquad (2.2)$$

Moreover if Λ' is a subgroup of an NEC group Λ of finite index, then Λ' is also an NEC group and the following Hurwitz-Riemann index formula holds

$$[\Lambda:\Lambda']=\mu(\Lambda')/\mu(\Lambda). \qquad (2.3)$$

Now since a surface group Γ of algebraic genus $\mathbf{p}$ has area $2\pi\alpha(\mathbf{p}-1)$, where $\alpha=2$ if Γ is a Fuchsian group (i.e. $k=0$ and the sign "+") and $\alpha=1$ otherwise, we see that the problem of finding a lower bound for the order of a group of automorphisms (lying in a certain family of finite groups $\mathscr{F}$) of a Klein surface of algebraic genus $\mathbf{p}\geq 2$ is equivalent to the problem of finding a lower bound for the area of NEC groups Λ that may admit a surface groups Γ with $\Lambda/\Gamma \in \mathscr{F}$. We shall refer to such a factor as to *$\mathscr{F}$-surface kernel (bordered or unbordered, orientable or nonorientable) factor* according as k>0 or k=0 and the sign is "+" or "-" in Γ. Having established such a bound, the problem of its realization is nothing else than the study of possible orders of factors Λ/Γ, where Λ is an NEC group with minimal possible area. The problem of the topological classification of corresponding Klein surfaces is equivalent to the problem of the the study possible signatures that a group Γ standing down in such quotient may have. As we shall see these problems are equivalent to purely group theoretical ones

In recent years a combinatorial method (based mainly on a chirurgy of fundamental regions of NEC groups) of determining algebraic structure of normal subgroups Λ' of an NEC group Λ in a function of the orders of the images in Λ/Λ' of the canonical generators of Λ and their certain products was developed in a series of articles **[2,3,4]**. The shortage of place does not give us the possibility of demonstrate how these results work, however it shall be pointed in the talk where we use it referring to it as to a *c-arguments* (combinatorial).

3. Orientable unbordered Klein surfaces.

This is the most classical case. Such a surface is just a Riemann surface and can be represented as D/Γ, where Γ is a Fuchsian surface group *i.e.* an NEC group with signature (g;+;[-];{-}).

In the recent paper **[9]** it was showed by means of c-arguments that a Fuchsian group Λ with signatures (0;+;[2,4,6];{-}),(0;+;[2,3,18];{-}) and areas $\pi/6$, $2\pi/9$ respectively are the ones with smallest area that may admit a Fuchsian surface group Γ as a normal subgroup with a supersoluble quotient Λ/Γ. Then it was showed that a first group admits only one such a factor (a semidirect product of a cyclic group of order 3 and a dihedral group of order 8). As a result using the Hurwitz Riemann formula we obtain that

A supersoluble group of automorphisms of a compact Riemann surface of genus $g\geq 3$ *has no more than* $18(g-1)$ *elements. A surface of genus* $g=2$ *can admit a supersoluble group of automorphisms of order* 24.

After, by purely group theoretic considerations, we showed that

A necessary and sufficient condition for the existence of a Riemann surface of genus $g\geq 3$ *that admits a supersoluble group of automorphisms of order* $18(g-1)$ *is that* 3^2 *divides* $g-1$ *and the only other (than 3) prime divisors of* $g-1$ *are congruent to* 1 mod 3.

It is worth to notice that some groups investigated here turned out to be subgroups of index 2 of some groups of automorphisms of bordered orientable Klein surfaces of maximal possible order that we shall present in the next section. This fact was one of the essential points in the proof of the above result.

4. Bordered orientable Klein surfaces.

Let $G=\Lambda/\Gamma$ be a bordered surface kernel factor. Then using c-arguments one can easily show that $\mu(\Lambda)\geq\pi/6$ and that the bound is attained only for an NEC group with signature $(0;+;[-];\{(2,2,2,3)\})$. Therefore from the Riemann Hurwitz formula we obtain that a bordered Klein surface of algebraic genus $\mathbf{p}\geq 2$ has at most $12(\mathbf{p}-1)$ elements. For obvious Reasons bordered Klein surfaces for which this bound is attained are said to have *maximal symmetry*. (The original proof of this result due to May **[16]** uses analytic arguments.) Moreover a finite group G can be represented as such factor if and only if it can be generated by three elements a,b and c of order 2 such that ab and ac have orders 2 and 3 respectively. In addition, if bc has order q then Γ have $k=|G|/2q$ empty period cycles and Γ is nonorientable if and only if ab and ac generate the whole group G. In particular we see that the problem of topological classification of bordered Klein surfaces admitting groups of automorphisms of maximal possible order is equivalent to the following purely group theoretic ones

Given $q\geq 1$ *find possible orders of groups admitting three generators* a,b *and* c *of order* 2 *such that* ab, ac *and* bc *have orders* 2,3 *and* q *respectively and then determine in which cases* G *can be generated by* ab *and* ac.

In general the above problem is very difficult, however its certain cases are more approachable. Recently May has solved it in **[17]** for the family of supersoluble groups obtaining the following results

(1) There are only two species of surfaces of algebraic genus **p**=2 *with maximal symmetry and supersoluble group of automorphisms of order* 12(**p**-1)=12; *a sphere with* 3 *holes and a torus with one hole.*
(2) The necessary and sufficient condition for the existence of a Klein surface X *of algebraic genus* **p**>2 *with maximal symmetry, supersoluble group of automorphisms of order* 12(**p**-1) *and with* k *boundary components is that*

(a) $\mathbf{p}-1=3^r$ *for some* $r\geq 1$,

(b) $k=3^s$ *for some integer* s *in range* $(r+1)/2\le s\le r$.

In such case X *is necessarily orientable.*

5. Bordered nonorientable Klein surfaces.

As we seen in in the previous section a bordered Klein surface of algebraic genus $p\ge3$ with maximal symmetry and supersoluble group of automorphisms is forced to be orientable. This leads us to ask for the bound of the order of a supersoluble group of automorphisms of bordered nonorientable Klein surface and for the topological classification of the corresponding species. Using c-arguments in the similar manner as in the previous sections we showed that this problem is equivalent to the following purely group theoretic ones

Find the smallest integer m *for which there exists a supersoluble group* G *generated by three elements* a,b,c *of order* 2 *such that* ab *and* ac *have orders* 2 *and* m *respectively and they generate the whole group* G.

Having solved the above problem determine possible orders of G, *and bc.*

Using certain properties of finite supersoluble groups we solved first problem, showing that m=6. This gives us the following result

A supersoluble group of automorphisms of a nonorientable bordered Klein surface X *of algebraic genus* $p\ge3$, *has at most* $6(p-1)$ *elements, whilst there exists a surface of algebraic genus* p=2 *(a real projective plane with two holes) having the dihedral group of order* 8 *(biggest possible) as a group of automorphisms*

The second problem turned out to be more difficult. Using rather deep combinatorial considerations concerning presentations of supersoluble groups by means of generators and relators we showed that

There exists a nonorientable Klein surface of algebraic genus $p \geq 3$ *having* k *boundary components and a supersoluble group* G *of automorphisms of order* $6(p-1)$ *if and only if*

(i) $p \equiv 3 \bmod (4)$

(ii) k *is a nontrivial power of* 3 *dividing* $6(p-1)$.

In such a case G *is an extension of a 3-group by a dihedral group.*

As we see a nonorientable Klein surface admitting a supersoluble group of the maximal possible order as a group of automorphisms has an odd algebraic genus unless $p=2$. This fact leads us to study surfaces of even algebraic genus. We have the following result

Let X *be a nonorientable Klein surface of even algebraic genus* $p \geq 2$ *and let* G *be a supersoluble group of automorphisms of* X. *Then* $|G| \leq 4p$. *This bound is attained for every even* p *and the corresponding Klein surface has* p *boundary components and so it is unique up to topological type.*

6. Nonorientable Riemann surfaces.

By a nonorientable Riemann surface we mean a nonorientable Klein surface without boundary. We prove recently **[10]** the following two results

A supersoluble group of automorphisms of a nonorientable Riemann surface of algebraic genus $p \geq 2$ *has at most* $12(p-1)$ *elements.*

A necessary and sufficient condition for the existence of a nonorientable Riemann surface of algebraic genus $p\geq 2$ *having a supersoluble group of automorphisms of order* $12(p-1)$ *is that* $p=3^n+1$ for some $n\geq 0$.

A group of automorphisms of a nonorientable Riemann surface without boundary can be viewed as a group of orientation-preserving automorphisms of an orientable Riemann surface of the same algebraic genus. Thus we see that the bound $18(p-1)$ (see section 3) for the order of a supersoluble group of automorphisms of compact Riemann surface cannot be attained in the nonorientable case. It is worth noting that this is in marked contrast with the corresponding results for the maximal groups of automorphisms of compact Klein surfaces for which the absolute bound $84(p-1)$ in the case of surfaces without boundary) and $12(q-1)$ (in the case with boundary) is attained both in orientable and in nonorientable cases (see for example [20] and [8] respectively)

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These notes collect some of the talks given in the Seminario del Departamento de Matemáticas Fundamentales de la U.N.E.D. in Madrid. Up to now the following titles have appeared:

1 **Luigi Grasselli,** Crystallizations and other manifold representations.
2 **Ricardo Piergallini**, Manifolds as branched covers of spheres.
3 **Gareth Jones**, Enumerating regular maps and hypermaps.
4 **J.C.Ferrando, M.López-Pellicer**, Barrelled spaces of class N and of class χ_o
5 **Pedro Morales**, Nuevos resultados en Teoria de la medida no conmutativa.
6 **Tomasz Natkaniec**, Algebraic structures generated by some families of real functions.
7 **Gonzalo Riera**, Algebras of Riemann matrices and the problem of units.
8 **Lynne D. James**, Representations of Maps.
9 **Grzegorz Gromadzki**, On supersoluble groups acting on Klein surfaces.
10 **Maria Teresa Lozano**, Flujos en 3-variedades.