UNIVERSIDAD NACIONAL DE EDUCACIÓN A DISTANCIA

DISERTACIONES DEL SEMINARIO DE MATEMÁTICAS FUNDAMENTALES

8

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REPRESENTATIONS OF MAPS

8

Representations of Maps

by

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This is the text of an address to the departamento de matemáticas fundamentales, U.N.E.D., Madrid, March 1990. It is an outline of a method of representing one topological category, for example, the category of cell decompositions of n-manifolds, by another, for example, the category of cell decompositions of oriented surfaces. This method also yields a non-commutative product of topological objects.

Representations of Maps

Lynne D. James

1. Introduction

We begin by outlining some examples of the well-known relationship between categories of topological objects, such as a cell decomposition of an n-manifold, algebraic objects, such as a conjugacy class of subgroups of a particular group, and combinatorial objects, such as an edge-coloured graph. See, for example, [BM], [BS], [CS], [G], [Jo], [JS], [La], [Li], [R], [S], [V_1], [V_2].

We then outline a method of representing one topological category by another. This method makes use of functors between the corresponding algebraic categories. In fact this idea can be seen as a generalization of the use of outer automorphisms of groups to induce operations on topological categories. Such operations can, in turn, be seen as a generalization of the well-known Poincaré duality which, for maps on surfaces, interchanges vertices and faces. See, for example, [Ja_1], [Ja_2], [JT], [LT], [W]. Two theorems are stated without proof and there follows a concrete example of a representation of the category of non-oriented 3-maps by the category of oriented maps on surfaces.

Finally, we show how the above method yields a non-commutative product of topological objects. Two theorems are stated without proof and there follows several concrete examples of products of topological objects.

2. Topology, Algebra and Combinatorics

2.1 Oriented Maps on Surfaces

We outline the theory of oriented maps on surfaces. A full account is given in [JS].

Let G be a connected graph, possibly with loops, multiple edges or free edges. Let S be an oriented surface without boundary. Then an <u>oriented map</u> M , on a surface, is an embedding of G in S , without crossings, such that the connected components of $S-G$ are homeomorphic to open discs. A <u>morphism</u> is a (possibly branched) covering of maps. This gives us a topological category.

Let Ω be the set of directed edges α of G . Let G be the group with presentation $\mathrm{G} = \langle x,y \mid x^2 = 1\rangle$. Then there is an action of G on Ω . The action of x is to change the direction of each directed edge. The action of y is to cyclically permute those directed edges pointing towards each vertex, according to the orientation of the map. See figure 1, where the orientation is anti-clockwise.

FIGURE 1

For example, if α is a directed edge which bounds a triangular face then $\alpha(xy)^3 = \alpha$. See figure 2, where the orientation is anti-clockwise.

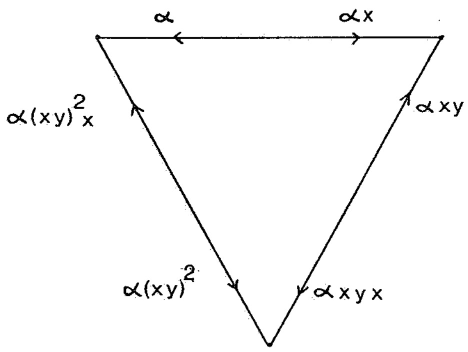


FIGURE 2

For a fixed $\alpha \in \Omega$ we define the map subgroup M_α to be the stabilizer of α in G. Varying the choice of $\alpha \in \Omega$ produces a conjugacy class of map subgroups. This gives us an algebraic category whose objects are conjugacy classes of subgroups M in G and whose morphisms are given by subgroup inclusions.

Finally, let Γ be the right coset graph of M_α in G with respect to the generators x and y . This gives us a combinatorial category whose objects are (isomorphism classes of) 2-coloured directed graphs Γ , such that the edges of the first colour are either loops or form pairs in the obvious way, with morphisms given by coverings of edge-coloured directed graphs.

The above topological, algebraic and combinatorial categories are equivalent. That is, they are related by invertible functors.

2.2 Non-Oriented Maps on Surfaces

We outline the theory of non-oriented maps on surfaces. A full account is given in [BS].

The theory is similar to that, given in 2.1, for the oriented case. However, now S is non-oriented (possibly non-orientable) and may have boundary. We allow the connected components of $S-G$ to be homeomorphic to either open discs or half discs.

Let Ω be the set of <u>blades</u>, or <u>2-flags</u>, of the map M. A blade may be thought of as an incidence of vertex, edge and face in M, or as a maximal simplex within the first barycentric subdivision of M. Let G be the group with presentation $G = \langle r,s,t \mid r^2=s^2=t^2=(rt)^2=1\rangle$. Then G is a Coxeter group with diagram $\underset{r}{\bullet}\overset{\infty}{—}\underset{s}{\bullet}\overset{\infty}{—}\underset{t}{\bullet}$. There is an action of G on Ω. The action of r, s, t is to change the vertex, edge, face, respectively, of $\alpha \in \Omega$. See figure 3.

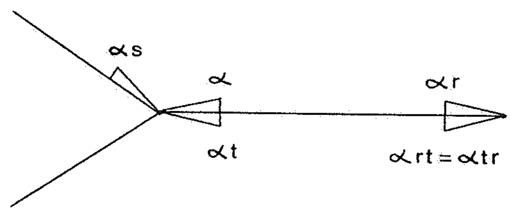


FIGURE 3

Again, by considering the stabilizers M_α in G of the blades $\alpha \in \Omega$, we obtain a conjugacy class of subgroups M in G, and thus an algebraic category.

We obtain a combinatorial category of 3-coloured graphs by considering the right coset graphs Γ of M_α in G with respect to the generators r, s and t.

2.3 Higher Dimensional Maps

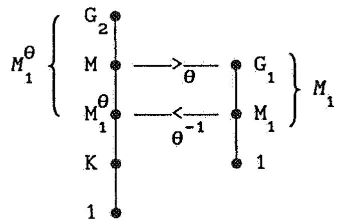
We simply remark that, in the same way as given in 2.2, it is possible to work with the category of n-maps, which includes the cell decompositions of n-manifolds, by considering the action of the Coxeter group G with diagram $\underset{0}{\bullet}\overset{\infty}{—}\underset{1}{\bullet}\overset{\infty}{—}\underset{2}{\bullet}\cdots\overset{\infty}{—}\underset{n}{\bullet}$ on the set Ω of n-flags of n-maps $\mathcal{M}$. See, for example, [Ja_1], [La], [R], [V_1].

3. Representations of Maps

3.1 Definitions

For $i = 1,2$ let $\mathfrak{M}_i$ be a category of objects $\mathcal{M}_i$, called maps, which correspond to conjugacy classes of subgroups M_i, called map subgroups, in a group G_i. For example, $\mathfrak{M}_1$ might be the category of oriented 2-maps (maps on surfaces) while $\mathfrak{M}_2$ might be the category of non-oriented 4-maps.

Now suppose we have a subgroup M in G_2 and an epimorphism $\theta : M \to G_1$. For each map $\mathcal{M}_1$ in $\mathfrak{M}_1$ corresponding to map subgroup M_1 in G_1 we define $\mathcal{M}_1^\theta$ to be the map in $\mathfrak{M}_2$ corresponding to map subgroup $\theta^{-1}(M_1)$ in G_2. We let M_1^θ denote $\theta^{-1}(M_1)$. Thus we have the following diagram



We now have a mapping, which we also denote by θ, $\theta : \{\mathcal{M}_1 | \mathcal{M}_1 \in \mathfrak{M}_1\} \to \{\mathcal{M}_1^\theta | \mathcal{M}_1 \in \mathfrak{M}_1\} \subseteq \mathfrak{M}_2$. We call θ a representation of $\mathfrak{M}_1$ by $\mathfrak{M}_2$.

3.2 Theorems

Our first theorem is little more than a remark and is stated without proof. It follows from the observation that the representation θ described in 3.1 is a functor.

THEOREM

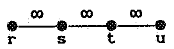
In the notation of 3.1, automorphisms and coverings of M_1 appear as automorphisms and coverings of M_1^{θ} .

Our second theorem follows from the observation that for any given $n \in \mathbb{N}$ there are infinitely many free subgroups in the Coxeter group of 2.2 of rank greater than n . It is stated without proof.

THEOREM

There are infinitely many ways of representing the category of n-maps by the category of 2-maps.

3.3 An Example

Let $\mathfrak{M}_1$ be the category of non-oriented 3-maps M_1 . Then M_1 corresponds to an algebraic object, namely a conjugacy class of subgroups M_1 in G_1 , where G_1 is the Coxeter group with diagram 

, and to a combinatorial object, namely the right coset graph Γ_1 of M_1 in G_1 with respect to the generators r, s, t and u .

Let $\mathfrak{M}_2$ be the category of oriented 2-maps M_2 . Then M_2 corresponds to an algebraic object, namely a conjugacy class of subgroups M_2 in G_2 , where G_2 is the group with presentation $G_2 = \langle x,y \mid x^2 = 1 \rangle$, and to a combinatorial object, namely the right coset graph Γ_2 of M_2 in G_2 with respect to the generators x and y .

Let $R = y^{-1}xy$, $S = yxy^{-1}$, $T = x$ and $U = y^3$. Let M be the subgroup of G_2 generated by R, S, T and U . Then M is a normal subgroup of index 3 in G_2 with presentation $M = \langle R,S,T,U | R^2=S^2=T^2=1 \rangle$. Moreover, there is an epimorphism $\theta : M \to G_1$ taking R, S, T, U to r, s, t, u respectively. This gives us a representation θ of $\mathfrak{M}_1$ by $\mathfrak{M}_2$ as described in 3.1.

3.4 A Construction

Given a non-oriented 3-map M_1 we show how to construct the representative oriented 2-map M_1^θ , where θ is the representation given in 3.3.

First note the action of R, S, T and U on the set of directed edges α of an oriented 2-map. See figure 4, where the orientation is anti-clockwise.

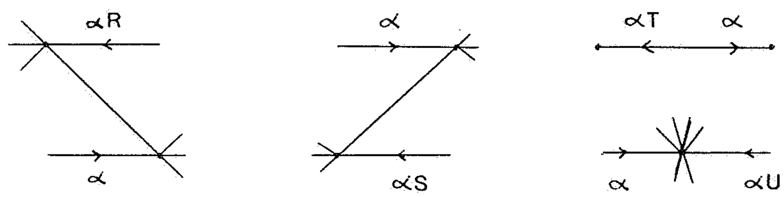


FIGURE 4

For each 3-flag α of M_1 there are three directed edges α^θ , $\alpha^\theta y$ and $\alpha^\theta y^{-1}$ of M_1^θ , one for each coset of M in G_2 . These directed edges lie in a square shaped piece of M_1^θ . See figure 5.

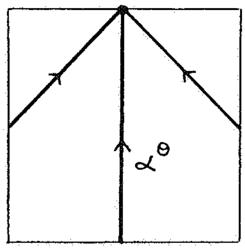


FIGURE 5

To construct M_1^θ we glue these pieces together using the rule $(\alpha w)^\theta = \alpha^\theta W$, where $w \in \{r,s,t,u\}$. Thus, for example, we have figure 6 as part of M_1^θ, where the orientation is anti-clockwise.

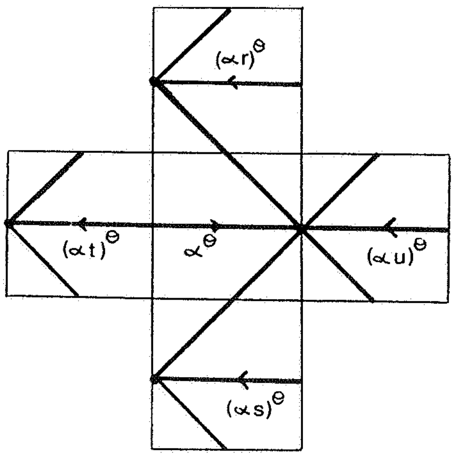


FIGURE 6

3.5 An Alternative Description

We give an alternative description of M_1^θ to that given in 3.4.

From figure 6 it is clear that there is a natural colouring by the set {r,s,t,u} of the edges of each square making up M_1^θ . If we split each vertex of M_1^θ then we obtain a closely related map. See figure 7, where the orientation is anti-clockwise.

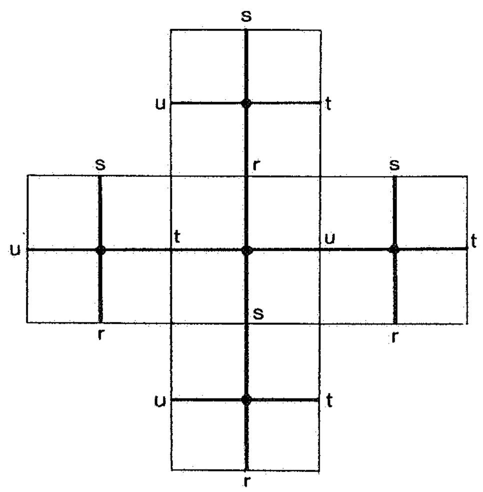


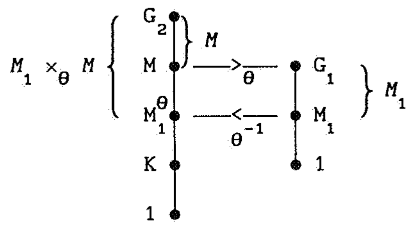
FIGURE 7

This is an embedding of a 4-coloured graph Γ . In fact Γ is the right coset graph Γ_1 described in 3.3. The embedding is according to the cyclic ordering (r,t,s,u). It may be interesting to compare figure 7 to [BM, Fig. 1]. We remark that there are many examples of representations which do not come from embedding the corresponding edge-coloured graphs.

4. Products of Maps

4.1 Definitions

In the notation of 3.1 we remark that a subgroup M in G_2 corresponds to an object M in $\mathfrak{M}_2$. For $M_1 \in \mathfrak{M}_1$ we define $M_1 \times_\theta M$ to be M_1^θ . Thus we have the following diagram.



4.2 Theorems

In the notation of 3.1 and 4.1, let Ω_1 be a set of right coset representatives for M_1 in G_1 . Then Ω_1 may be identified with a set of right coset representatives for M_1^θ in M . We let Ω be a set of right coset representatives for M in G_2 . Then $\Omega_1 \times \Omega$ may be identified with a set of right coset representatives for M_1^θ in G_2 . Let $\pi_1 : G_1 \to S(\Omega_1)$, $\pi : G_2 \to S(\Omega)$ and $\pi_2 : G_2 \to S(\Omega_1 \times \Omega)$ be the transitive permutation representations corresponding to the actions, by right multiplication, on the right cosets of M_1 , M and M_1^θ in G_1 , G_2 and G_2 respectively. These actions correspond to the maps M_1 , M and M_1^θ respectively.

The following theorem is an attempt to justify the adopted product notation. It is stated without proof.

THEOREM

Here the notation is as above.

Firstly, if we fix the second coordinate of $\Omega_1 \times \Omega$ to that which represents the coset M in G_2 then π_2 restricts to an action of M on the first coordinate which is equivalent to $\pi_1 \circ \theta$, and so corresponds to the map $\mathcal{M}_1$ with an element of twist.

Secondly, the action π_2 restricts to an action of G_2 on the second coordinate which is equivalent to π, and so corresponds to the map $\mathcal{M}$.

The following theorem follows from the observation that M_1^{θ} is a subgroup of M.

THEOREM

$\mathcal{M}_1 \times_\theta \mathcal{M}$ covers $\mathcal{M}$.

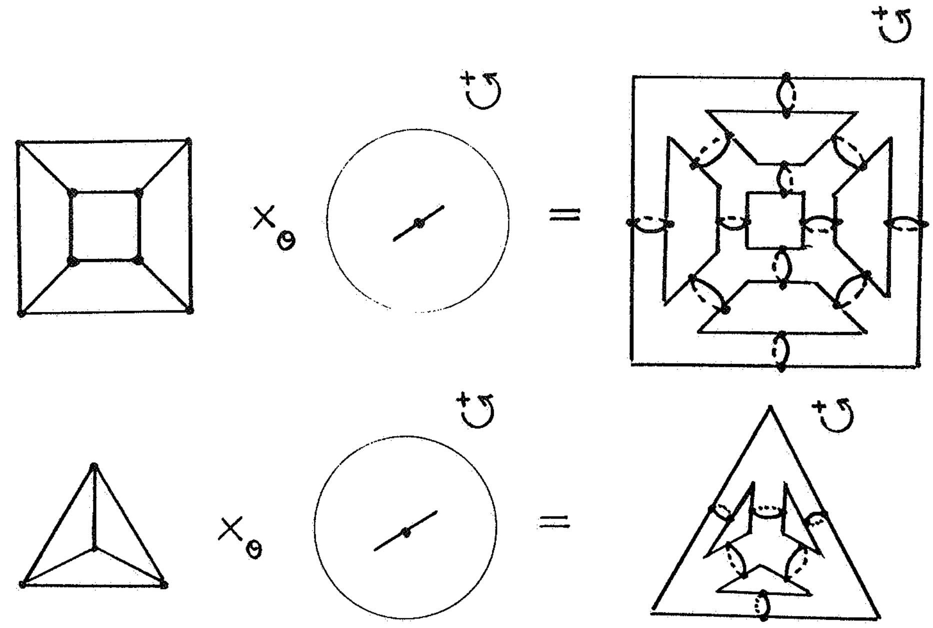
4.3 Examples

The aim of this section is simply to give the reader a flavour of the product defined in 4.1. We omit a large amount of detail, and simply claim that in each given example there exists a representation θ which yields the stated product(s).

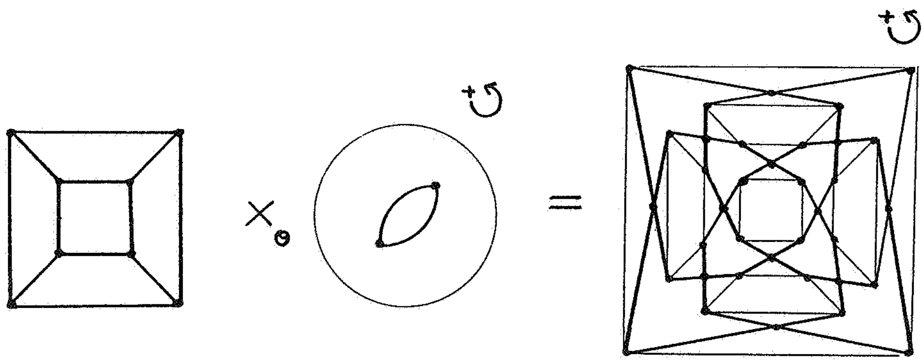
Recalling the first theorem of 3.2 we observe that the first factor of the first product in example 1 is a non-oriented cube whose automorphism group thus includes reflections and is thus isomorphic to $S_4 \times C_2$. This group should then appear as a group of automorphisms of the product, which is an oriented map on a surface of genus 5, and whose automorphisms are necessarily orientation-preserving. Recalling the second theorem of 4.2 we observe that in each example the product map should cover the second factor.

Example 4 is the product yielded by the representation given in 3.3. Example 5 is included to demonstrate the factorization of a given map, in this case an oriented tetrahedron, and the problem of finding irreducible factors. The stated factorization was obtained by considering the stabilizer of a point in the natural action of A_4, and the complementary Klein-4 group. Example 6 is included to demonstrate how Poincaré duality arises as a special case. Example 7 is included to demonstrate how the standard representation of a map by a hypermap arises as a special case. For an account of hypermaps see, for example, [Ja_2].

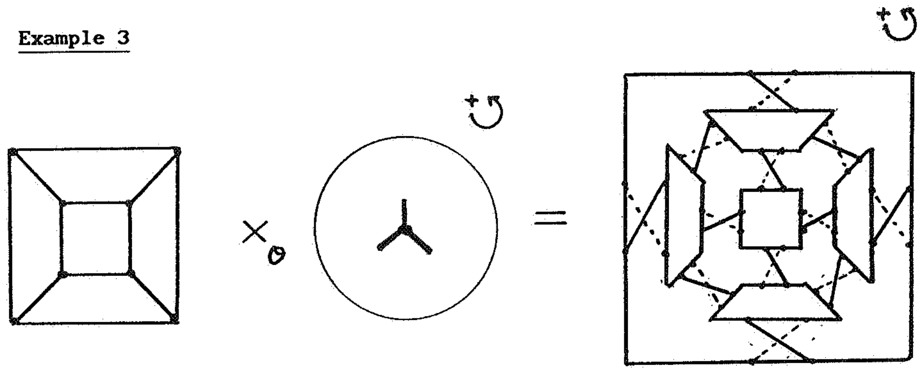
Example 1



Example 2

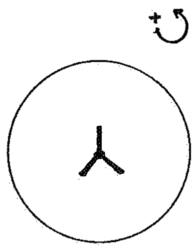


Example 3



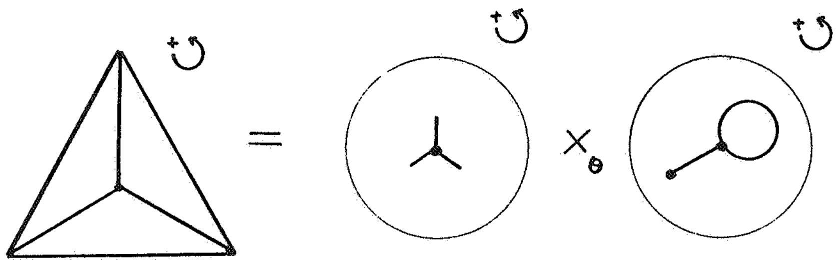
Example 4

Non-oriented 3-maps $\times_\theta$

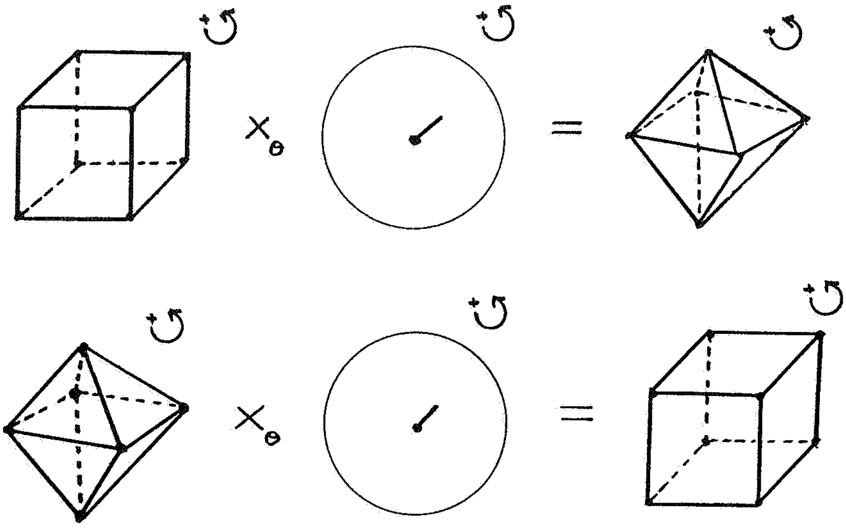


$\subseteq$ Oriented 2-maps

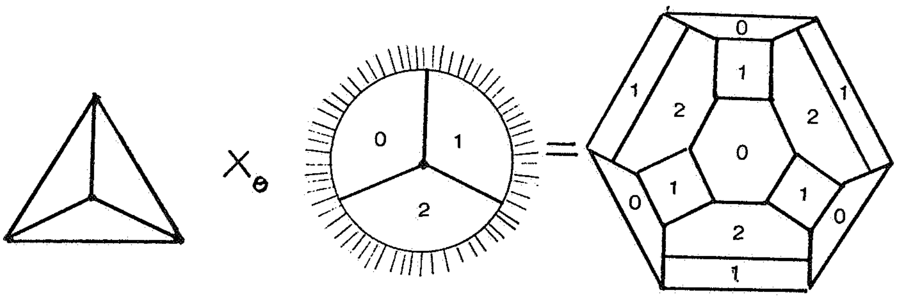
Example 5



Example 6



Example 7



5. Concluding Remarks

In the notation of 3.1, given a subgroup M in G_2, epimorphisms $\theta : M \to G_1$ induce functors, or representations, $\theta : \mathfrak{M}_1 \to \mathfrak{M}_2$ taking a map $\mathcal{M}_1$ corresponding to a subgroup M_1 in G_1 to a map $\mathcal{M}_1^{\theta}$ corresponding to the subgroup $\theta^{-1}(M_1)$.

These are not the only examples of functors $\theta : \mathfrak{M}_1 \to \mathfrak{M}_2$. Clearly, given a subgroup M in G_2, epimorphisms $\theta : G_1 \to M$ induce functors, or representations, $\theta : \mathfrak{M}_1 \to \mathfrak{M}_2$ taking a map $\mathcal{M}_1$ corresponding to a subgroup M_1 in G_1 to a map $\mathcal{M}_1^{\theta}$ corresponding to the subgroup $\theta(M_1)$, and such representations behave rather differently to the ones described in this paper.

It remains an open problem to fully classify all functors $\theta : \mathfrak{M}_1 \to \mathfrak{M}_2$ between the sorts of categories discussed in this paper.

References

[BM] Brancho, J. and Montejano, L., 'The combinatorics of colored triangulations of manifolds', Geometriae Dedicata, 22 (1987), 303-328.

[BS] Bryant, R.P. and Singerman, D., 'Foundations of the theory of maps on surfaces with boundary', Quart. J. Math. Oxford (2), 36 (1985), 17-41.

[CS] Corn, D. and Singerman, D., 'Regular hypermaps', Europ. J. Combinatorics, 9 (1988), 337-351.

[G] Gagliardi, C., 'A combinatorial characterization of 3-manifold crystalizations', Bollitino U.M.I. (5), 16 - A (1979), 1-9.

[Ja_1] James, L.D., 'Complexes and Coxeter groups - operations and outer automorphisms', J. Algebra, 113 (1988), 339-345.

[Ja_2] James, L.D., 'Operations on hypermaps, and outer automorphisms', Europ. J. Combinatorics, 9 (1988), 551-560.

[Jo] Jones, G.A., 'Graph imbeddings, groups, and Riemann surfaces', Colloquia Mathematica Societatis. János Bolyai 25, Algebraic Methods in Graph Theory, Szeged (Hungary), (1978), 297-311.

[JS] Jones, G.A. and Singerman, D., 'Theory of maps on orientable surfaces', Proc. London Math. Soc. (3), 37 (1978), 273-307.

[JT] Jones, G.A. and Thornton, J.S., 'Operations on maps and outer automorphisms', J. Combin. Theory Ser. B, 35 (1983), 93-103.

[La] Lannér, F., 'On complexes with transitive groups of automorphisms', Medd. Lunds. Univ. Maths., 11 (1950), 1-71.

[Li] Lins, S., 'Graph-encoded maps', J. Combin. Theory Ser. B, 32 (1982), 171-181.

[LT] Léger. C. and Terrasson, J.-C., 'L'action du groupe symétrique S3 sur l'ensemble des classes d'isomorphisme de pavages de surface fermée', C.R. Acad. Sc. Paris, t. 302, Série I, 1 (1986), 39-42.

[R] Ronan, M.A., 'Coverings and automorphisms of chamber systems', Europ. J. Combinatorics, 1 (1980), 259-269.

[S] Singerman, D., 'Maps on surfaces with boundary', Ars Combinatoria, Vol. 16-A (1983), 227-238.

[V_1] Vince, A., 'Combinatorial maps', J. Combin. Theory Ser. B, 34 (1983), 1-21.

[V_2] Vince, A., 'Regular combinatorial maps', J. Combin. Theory Ser. B, 35 (1983), 256-277.

[W] Wilson, S.E., 'Operators over regular maps', Pacific Jnl. Maths., 81 (1979), 559-568.

These notes collect some of the talks given in the Seminario del Departamento de Matemáticas Fundamentales de la U.N.E.D. in Madrid. Up to now the following titles have appeared:

1 **Luigi Grasselli,** Crystallizations and other manifold representations.
2 **Ricardo Piergallini**, Manifolds as branched covers of spheres.
3 **Gareth Jones,** Enumerating regular maps and hypermaps.
4 **J.C.Ferrando, M.López-Pellicer**, Barrelled spaces of class N and of class χ_0
5 **Pedro Morales**, Nuevos resultados en Teoria de la medida no conmutativa.
6 **Tomasz Natkaniec**, Algebraic structures generated by some families of real functions.
7 **Gonzalo Riera**, Algebras of Riemann matrices and the problem of units.
8 **Lynne D. James**, Representations of Maps.
9 **Grzegorz Gromadzki,** On supersoluble groups acting on Klein surfaces.
10 **Maria Teresa Lozano**, Flujos en 3-variedades.