UNIVERSIDAD NACIONAL DE EDUCACIÓN A DISTANCIA

DISERTACIONES DEL SEMINARIO DE MATEMÁTICAS FUNDAMENTALES

7

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ALGEBRAS OF RIEMANN MATRICES AND THE PROBLEM OF UNITS

7

Algebras of Riemann matrices and the problem of units

by

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In a paper written in 1934 (*c.f.* [1]) H.Weyl considers the correspondences of a Riemann surface into itself as associated to some algebras of matrices. He does so in an unusual way since instead of the matrix of periods Z on a surface he considers a real matrix R $2g \times 2g$ formally equivalent to the former but much more suited to algebraic considerations. Latter C.L.Siegel (*c.f.* [2]). completed the study in much the same fashion but leaving open a number of questions that we shall explain.

Riemann Matrices.

Consider a basis of closed curves $\alpha_1,\ldots,\alpha_{2g}$ in $H_1(M,\mathbb{Z})$, where M is a compact Riemann surface of genus g. A basis of analytic differentials $dw_1,\ldots,$ dw_{2g} in $H^{1,0}(M,\mathbb{C})$ is said to be *dual to the basis of curves* if

$$Re \int_{\alpha_j} dw_i = -\alpha_i \cdot \alpha_j = c_{ij},$$

where $\alpha \cdot \beta$ denote the intersection product.

In this case the Riemann relations imply that the matrix

$$Im \int_{\alpha_j} dw_i = s_{ij}$$

is symmetric positive definite. Multiplication by i in the vector space of differentials is represented in terms of the basis (dw_i) *by a real matrix* R such that $R^2 = -I$.

We have the relations

$$R = C^1 S, \quad C' = -C, \quad S' = S,$$

where $'$ is the transposition.

In other words, the matrix R defines a complex structure in the real vector space $\mathbb{R}^{2g}$ and the matrices C, S define an hermitian scalar product by the formula

$$<u,v> = u'Sv - iu'Cv.$$

Example: g = 1.

M is given as the quotient of $\mathbb{C}$ by the lattice generated by 1 and $\tau = r + it,\ t > 0$. These two vectors form a basis of $H_1(M,\mathbb{Z})$ and we have

$$dw_1 = (i/t)dz,\ dw_2 = (1 + ir/t)dz$$

since, for instance,

$$Re \int_\tau dw_1 = Re(-1 + ir/t) = -1 = -(1.\tau).$$

Here then

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\quad S = \begin{bmatrix} 1/t & r/t \\ r/t & (r^2+t^2)/t \end{bmatrix},\quad R = \begin{bmatrix} r/t & (r^2+t^2)/t \\ -1/t & -r/t \end{bmatrix}.$$

This example can be generalized in much the same way to any Riemann surface with a canonical basis.

Correspondences.

An (m,n) correspondence between two surfaces M, N associates to each point p in M n transforms $p_1,\ldots,p_n$ in N varying holomorphically with p and to each point q in M m transforms $q_1,\ldots,q_m$ in M varying holomorphically with q.

To give a more precise definition consider the surface $M \times N$. If M and N are given as curves in $\mathbb{P}_k$ and $\mathbb{P}_l$ with points $(x_0,\ldots,x_k)$ in M and $(y_0,\ldots,y_l)$ in N then the points

$$\rho\ x_i\ y_j$$

define the product surface in $\mathbb{P}_{(k+1)(l+1)-1}$.

The surface $M \times N$ carries two pencils of curves $M \times n_0$, $m_0 \times N$ with algebraic intersection 1. A correspondence is then defined as an algebraic curve $\Gamma \subseteq M \times N$. To each point p in M the transforms are given as

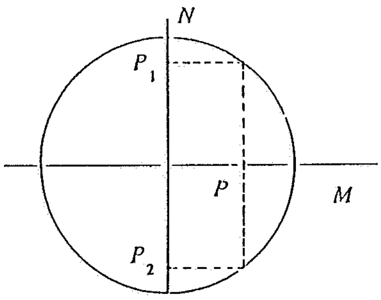
$$p \times M \ . \ \Gamma = p_1 + \dots + p_n \text{ in } N$$

and for each point q in N we have

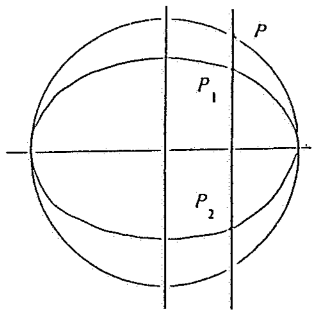
$$\Gamma \ . \ q \times M = q_1 + \dots + q_n \text{ in } M.$$

Examples.

(1) The circle Γ: $x^2 + y^2 = 1$ in affine coordinates defines a $(2,2)$ correspondence from $\mathbb{P}_1$ to $\mathbb{P}_1$.



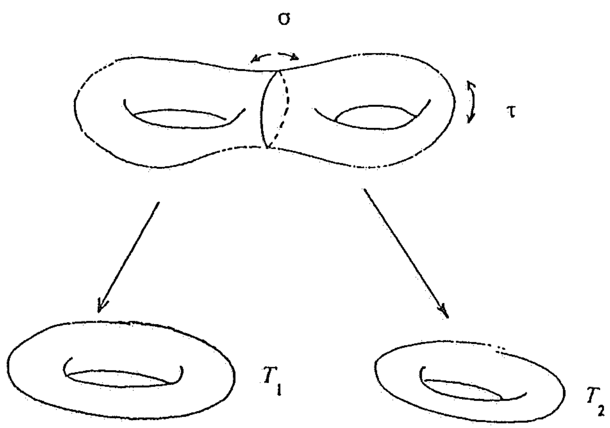
(2) The lines $x = constant$ define a $(2,2)$ correspondence from a circle to an elipse



$$x^2 + y^2 = 1$$

$$k^2x^2 + y^2 = 1$$

(3) Consider a surface C of genus 2 with an automorphism σ; $\sigma^2 = id$ and $C/\langle\sigma\rangle = T_1$ is of genus 1. Then if τ is the hyperelliptic involution $C/\langle\sigma\tau\rangle = T_2$. The surface C defines in a natural way a $(2,2)$ correspondence from T_1 to T_2.



(4) Let Γ be a discrete Fuchsian group acting on the unit disc $\mathbb{D}$ and Γ_1 Γ_2 be two Fuchsian groups such that $[\Gamma : \Gamma_1] = n$, $[\Gamma : \Gamma_2] = m$. Then the identity in $\mathbb{D}$ gives an (n,m) correspondence from $\mathbb{D}/\Gamma_1$ to $\mathbb{D}/\Gamma_2$.

Or, let $\mathbb{D}/\Gamma$ admit a group of automorphisms G. For two subgroups H_1, H_2 in G we have a correspondence from $(\mathbb{D}/\Gamma)/H_1$ to $(\mathbb{D}/\Gamma)/H_2$.

Algebras of Matrices.

Let Γ be a correspondence from M to M. As p varies over a cycle α_i in $H_1(M,\mathbb{Z})$ the n points $p_1 + \ldots + p_n$ together vary over a cycle

$$\Gamma_*(\alpha_i) = \sum_{j=1}^{2g} a_{ji}\alpha_j\ , \quad a_{ji} \in \mathbb{Z}.$$

Thus Γ is represented by a matrix $A : H_1(M,\mathbb{Z}) \longrightarrow H_1(M,\mathbb{Z})$ with integer coefficients.

Theorem (*Lefchetz*).

A necessary and sufficient condition for an integral matrix A to be given by an algebraic correspondence Γ is that $AR = RA$.

(Actually Lefchetz did not prove this theorem in this language but this is a translation of his theorem to decide when a cycle Γ is algebraic in $M \times N$)

Theorem (*Torelli*).

A necessary and sufficient condition for an integral matrix $\pm A$ *such that* $AR = RA$ *to be the matrix of an isomorphism* $\tau : M \longrightarrow M$ *is that* $AA^* = I$, *where* $A^* = C^{-1}A'C$.

(Actually Torelli's theorem is not written in this way either but it is its interpretation in this language.

H.Weyl then posses the problem:

Classify all algebras of matrices A that appear as algebras of matrices commuting with a given R.

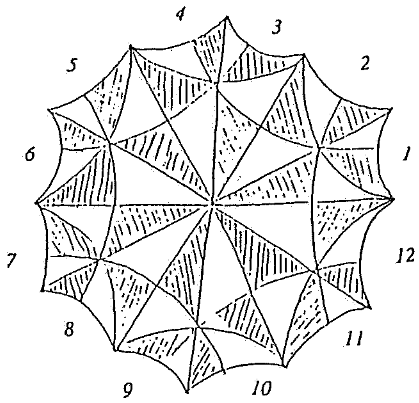
This work can be read in Siegel's lectures (*c.f.* [2]). But we have two main problems

(a) The matrices A have integral coefficients. We have to consider then "orders" and not just algebras.

(b) The automorphisms τ correspond to units, or to integral elements in the algebra such that $a \cdot a^* = 1$, where a^* is an involution. How do we find them? What are the possible R for a given genus that admit non-trivial units? We know for a fact that they can be classified geometrically or as algebraic curves.

Examples.

(1) Consider the curve of genus *2* admitting a group of automorphisms of order *24*. The normalizer is generated by reflections in a triangle of interior angles $\pi/6$, $\pi/2$, $\pi/4$.



1 - 4
3 - 6
5 - 8
7 - 10
9 - 12
11 - 2

$g = 2$

Basis $\alpha_1 = 2 + 3$, $\alpha_2 = -1 + 5$, $\alpha_3 = -3 + 7$, $\alpha_4 = -5 + 9$. In terms of this basis we have for the two generators of order 6 and 4 respectively the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad A^6 = I,\ A^4 + A^2 + I = 0$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \qquad B^4 = I,\ B^2 + I = 0$$

or, under a change of coordinates that brings

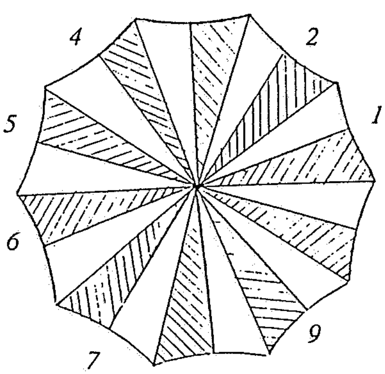
$$C = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{to} \quad \tilde{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & =1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

R has to satisfy, $\tilde{A}R = R\tilde{A}$, $\tilde{B}R = R\tilde{B}$, $R^2 = -I$ and then we find

$$\tilde{R} = \sqrt{\frac{1}{3}} \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ -2 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix}$$

(2) Consider the curve of genus 2 admitting a group of automorphisms of order 10 ($\mathbb{Z}_{10}$).



A basis of closed curves is $\alpha_1 = 1 + 2, \alpha_2 = 2 + 3, \alpha_3 = 3 + 4, \alpha_4 = 4 + 5.$

The rotation of order 10 is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad A^{10} = I,\ A^4 - A^3 + A^2 - A + I = 0$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

To find R such that $AR = RA$ is not altogether trivial. Namely one can check that if R commutes with A then it has to be in $\mathbb{R}[A]$, that is $R = a_0 I + a_1 A + a_2 A^2 + a_3 A^3$ with $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Hence we have to look for solutions of $R^2 = -I$ in the algebra

$\mathbb{R}[A] = \mathbb{R}[x]/(x^4 - x^3 + x^2 - x + 1) =$

$$\mathbb{R}[x]/(x^2-2\cos(2\pi/5)x+1) \oplus \mathbb{R}[x]/(x^2-2\cos(3\pi/5)x+1) \cong \mathbb{C} \oplus \mathbb{C}$$

as algebras. On the last term we find four such solutions $(\pm i,\pm i)$ and tracing back the isomorphisms we find four such R's. Now the condition CR positive definite implies

$$R = (\sqrt{5})/2)\sqrt{(5+\sqrt{5})/2}\begin{bmatrix} -(3+\sqrt{5})/4 & -(1+\sqrt{5})/2 & (1-\sqrt{5})/2 & -(1+\sqrt{5})/2 \\ (3+\sqrt{5})/2 & (\sqrt{5}-1)/4 & -1 & 1 \\ -1 & 1 & (1-\sqrt{5})/4 & -(3+\sqrt{5})/2 \\ (1+\sqrt{5})/2 & (\sqrt{5}-1)/2 & (1+\sqrt{5})/2 & (3+\sqrt{5})/4 \end{bmatrix}$$

The problem we consider in these examples are the following :

- Characterize the integral elements in the algebra of all matrices commuting with R
- Characterize the units in these algebras and prove that we obtain all of the automorphisms back.
- Understand why there are only 3 such algebras in genus 2. (These two and the last one with a group of order 48).

References

1 H. Weyl, On generalized Riemann matrices, Annals of Mathematics **35**(4), 1934, 714-729.

2 C.L. Siegel, Lectures on Riemann Matrices, Tata Lecture Notes, 1963.

These notes collect some of the talks given in the Seminario del Departamento de Matemáticas Fundamentales de la U.N.E.D. in Madrid. Up to now the following titles have appeared:

1 **Luigi Grasselli,** Crystallizations and other manifold representations.
2 **Ricardo Piergallini**, Manifolds as branched covers of spheres.
3 **Gareth Jones**, Enumerating regular maps and hypermaps.
4 **J.C.Ferrando, M.López-Pellicer**, Barrelled spaces of class N and of class χ_0
5 **Pedro Morales**, Nuevos resultados en Teoria de la medida no conmutativa.
6 **Tomasz Natkaniec**, Algebraic structures generated by some families of real functions.
7 **Gonzalo Riera**, Algebras of Riemann matrices and the problem of units.
8 **Lynne D. James**, Representations of Maps.
9 **Grzegorz Gromadzki**, On supersoluble groups acting on Klein surfaces.
10 **Maria Teresa Lozano**, Flujos en 3-variedades.