UNIVERSIDAD NACIONAL DE EDUCACIÓN A DISTANCIA

DISERTACIONES DEL SEMINARIO DE MATEMÁTICAS FUNDAMENTALES

6

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ALGEBRAIC STRUCTURES GENERATED BY SOME FAMILIES OF REAL FUNCTIONS

6

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by

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1. Definitions. R denotes the real line. For a given .set X let R^X be the family of all real-valued functions defined on X. We can consider the following operations defined on R^X.

(A) *Algebraic operations.*

$$\forall\ f, g \in R^X \qquad f+g : x \longmapsto f(x)+g(x), \qquad f\cdot g : x \longmapsto f(x)\cdot g(x),$$

$$\forall\ f \in R^X \quad \forall\ a \in R \qquad (a\cdot f) : x \longmapsto a\cdot(f(x)).$$

The set R^X with such defined operations forms an algebra of functions (over R). If $Y \subseteq R^X$ then $\mathcal{A}(Y)$ denotes the subalgebra of R^X generated by Y. Of course, in general $\mathcal{A}(Y)$ may be different from Y.

(B) *Operations of lattice.*

$$\forall\ f, g \in R^X \qquad max(f,g) : x \longmapsto max(f(x),g(x)),$$

$$min(f,g) : x \longmapsto min(f(x),g(x)).$$

If $Y \subseteq R^X$ then $\mathcal{L}(Y)$ denotes the lattice generated by Y, *i.e.* the smallest lattice of functions containing Y.

(C) *The Baire system.*

For $Y \subseteq R^X$ let $\mathcal{B}(Y)$ denotes the Baire closure of Y, *i.e.* the collection of all pointwise limits of sequences taken from Y

$$B(Y) = \{f : X \longrightarrow R \mid \exists\ (f_n)_{n\in N}\ f_n \in Y \text{ and } f : x \longmapsto \lim_{n\to\infty} f_n(x)\}.$$

For every ordinal $\alpha<\omega_1$ we define the family $\mathcal{B}_\alpha(Y)$ as follows:

$$\mathcal{B}_\alpha(Y) = B(\bigcup_{\beta<\alpha} \mathcal{B}_\beta(Y)).$$

Finally $$\mathcal{B}(Y) = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha(Y).$$

Example. If $X = R$, $Y = \mathcal{C}$ is the family of all continuous functions $f : R \longrightarrow R$, then $\mathcal{A}(\mathcal{C}) = \mathcal{C}$, $\mathcal{L}(\mathcal{C}) = \mathcal{C}$ and $\mathcal{B}_\alpha(\mathcal{C})$ is the family of all functions of the Baire class α.

Many mathematicians have studied some families of functions, and algebras and Baire systems generated by those families (*e.g.* Kuratowski, Sierpinski, Preiss, Mauldin, Miller). In Bydgoszcz we have studied also lattices of functions.

2. One-to-one functions. Let $\mathcal{R}_1$ be a family of all one-to-one functions $f : R \longrightarrow R$.

(A) It is easy to see that every function $f : R \longrightarrow R$ is a sum of two one-to-one functions. Indeed, list every reals in the sequence $(x_\xi)_{\xi<\mathfrak{c}}$. We define (inductively) two one-to-one functions $f_1, f_2 : R \longrightarrow R$ such that:

$$f_1(x_\alpha) \in R \setminus \left\{ \{f_1(x_\beta) \mid \beta<\alpha\} \cup \{f(x_\alpha) - f_2(x_\beta) \mid \beta<\alpha\} \right\}$$

and

$$f_2(x_\alpha) = f(x_\alpha) - f_1(x_\alpha) \text{ for each } \alpha < \mathfrak{c}.$$

Then f_1 and f_2 are one-to-one and $f = f_1 + f_2$. Thus $\mathcal{A}(\mathcal{R}_1) = R^R$.

(B) Similarly we can prove that every function $f : R \longrightarrow R$ is a pointwise limit of some sequence of one-to-one functions. Thus $\mathcal{B}(\mathcal{R}_1) = \mathcal{B}_1(\mathcal{R}_1) = R^R$.

(C) A little more complicated situation is in the case of the lattice generated by the one-to-one functions **[5]**. First we introduce the family

$$\mathcal{R} = \{ f : R \longrightarrow R \mid \exists\, n \in N \ \ \forall y \in R \ \ card(f^{-1}(y)) \le n \}.$$

For every $f \in \mathcal{R}$ we define $n_0(f) = max\{n \mid \exists y \in R \ card(f^{-1}(y)) = n\}$. Let finally $\mathcal{R}_n = \{ f \in \mathcal{R} \mid n_0(f) \le n \}$. Then $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$ and we can prove (by induction) that:

(i) if $f, g \in \mathcal{R}_n$ then $max(f,g) \in \mathcal{R}_{2n}$ and $min(f,g) \in \mathcal{R}_{2n}$,

(ii) every function $f \in \mathcal{R}_n$ can be expressed as a maximum of four functions from the class $\mathcal{R}_{n-1}$. (Observe that in this realization of f appears only the operation of maximum; the operation of minimum is not necessary)

Consequently we obtain that $\mathcal{L}(\mathcal{R}_1) = \mathcal{R}$.

3. Differentiable functions. Let *Diff* be the family of all differentiable functions. Of course *Diff* is an algebra of functions, $\mathcal{A}(Diff) = Diff$ and it is well-known that $B(Diff) = B(\mathcal{C}) = \mathcal{B}_1$. Evidently *Diff* is not a lattice and Z. Grande posed in **[7]** the problem to characterize the lattice generated by differentiable functions .

In **[11]** we proved that $\mathcal{L}(Diff) = \mathcal{S}$, where $\mathcal{S}$ is the family of all continuous functions $f : R \longrightarrow R$ such that

(1) the set $N(f)$ of all points at which f is not differentiable is a finite union of discrete sets,

(2) for every point $x \in R$ there exist the right-hand derivative $f'_+(x)$ and the left-hand derivative f'_- at this point.

It is easy to see that $Diff \subseteq \mathcal{S}$,$\mathcal{S}$ is a lattice of functions and therefore, $\mathcal{L}(Diff) \subseteq \mathcal{S}$. The proof of the inclusion $\mathcal{S} \subseteq \mathcal{L}(Diff)$ is based on the following lemma.

Lemma. *For $A \subseteq R$ let $der(A)$ denote the set of all accumulation points of A which belong to A. Also let*

$$der^0(A)=A \quad \text{and} \quad der^{k+1}(A)=der(der^k(A)).$$

Then for every $C \subseteq R$ and for every $n=0,1...,$ the following are equivalent:

(i) C is a union of n discrete sets,

(ii) $der^n(C)=\emptyset$.

The idea of the proof that $\mathcal{S} \subseteq \mathcal{L}(Diff)$ looks as follows. We define

$$\mathcal{S}_0 = Diff \quad \text{and} \quad \mathcal{S}_n = \{ f \in \mathcal{S} \mid der^n(N(f)) = \emptyset \}.$$

Then $\mathcal{S} = \bigcup_{n \in N} \mathcal{S}_n$ and we can prove that for every function $f \in \mathcal{S}_n$ there exist functions $f_1, f_2, f_3, f_4 \in \mathcal{S}_{n-1}$ such that

$$f = min(max(f_1, f_2), max(f_3, f_4)).$$

4. Darboux functions and almost continuity. Recall that a function $f : R \longrightarrow R$ has the *Darboux property* iff for every connected subset C of R, $f(C)$ is a connected subset of R. A function $f : R \longrightarrow R$ is said to be *almost continuous* (in the sense of Stallings **[16]**) if for every open set $G \subseteq R \times R$ containing f, there exists a continuous function $g : R \longrightarrow R$ lying entirely in G.

Let $\mathcal{C}$ (*resp.* $\mathcal{A}$) denote the class of all continuous functions, (*resp.* almost continuous functions) and let $\mathcal{D}$ be the family of all Darboux functions. We have $\mathcal{C} \subsetneqq \mathcal{A} \subsetneqq \mathcal{D}$ (see *e.g.***[2]**).

The following theorem described the algebra, the lattice and the Baire system generated by the family of all almost continuous functions (and consequently, Darboux functions).

Theorem 1. *Let $f : R \longrightarrow R$ be any function. Then*

(a) $f = f_1 + f_2$ for some functions $f_1, f_2 \in \mathcal{A}$ **[10]**,

(b) f is a pointwise limit of some sequence of almost continuous functions: $f = \lim f_i$, $f_i \in \mathcal{A}$ **[10]**,

(c) $f = min(max(f_1, f_2), max(f_3, f_4))$ for some $f_1, f_2, f_3, f_4 \in \mathcal{A}$ **[12]**.

Moreover, if h is Lebesgue measurable or with the Baire property then the function f_i in (a), (b), (c) may be taken to be Lebesgue measurable or with the Baire property (**[8]** *and* **[14]**).

Remark. We have also the stronger version of (c) :
$f = max(t_1, t_2)$, where $t_1 = min(max(f_1, f_2), f_3)$, $t_2 = min(max(f_1, f_3), f_2)$ and f_1, f_2, f_3 are almost continuous, but there exist functions $f : R \longrightarrow R$ such

that $f \neq max(f_1,f_2)$ and $f \neq min(f_1,f_2)$ for each $f_1, f_2 \in \mathcal{D}$ (and for each $f_1, f_2 \in \mathcal{A}$ too). The function $f(x)=x$ for $x \in \{1,-1\}$ and $f(x)=0$ otherwise is an example of such a function.

Theorem 2. *Let $\mathcal{DB}_\alpha$ $(\alpha \geq 1)$ denote the class of all functions of the Baire class α with the Darboux property. Then*

(a) for every $f \in \mathcal{B}_\alpha$ there exist functions $f_1, f_2 \in \mathcal{DB}_\alpha$ such that $f = f_1 + f_2$ **[3]**,

(b) for every $f \in \mathcal{B}_{\alpha+1}$ there exists a sequence of functions $(f_n)_n$, $f_n \in \mathcal{DB}_\alpha$ such that f is a pointwise limit of this sequence **[3]**,

(c) for every $f \in \mathcal{B}_\alpha$ there exist functions $f_1, f_2, f_3, f_4 \in \mathcal{DB}_\alpha$ such that $f = min(maxf_1, f_2),max(f_3, f_4))$ **[12]**.

Since $\mathcal{DB}_1 = \mathcal{AB}_1$ **[1]**, the same theorem holds for the class $\mathcal{AB}_1$.

Problem 1. Does the analogous theorem for the class $\mathcal{AB}_\alpha$, where $\alpha>1$ hold ?

J. Ceder characterized those functions which are products of Darboux functions **[4]**. This are all functions $f : R \longrightarrow R$ such that f has a zero in each interval in which f changes sign. A function $f : R \longrightarrow R$ is a quotient of two Darboux functions if f satisfies the following conditions:

(i) if $a < b$ and $f(a) \cdot f(b) < 0$ then $f(c) = 0$ for some $c \in (a,b)$ and

(ii) the sets $[f > 0]$ and $[f < 0]$ are bilaterally c-dense in themselves **[13]**.

Problem 2. Characterization of functions which are products and quotients of almost continuous functions.

It is easy to see that the class of all Darboux functions is closed with respect to the superposition of functions. This is not true for almost continuous functions; every function $f : [0,1] \longrightarrow [0,1]$ which take every value c–times in every subinterval, is a superposition of two almost continuous functions g, $h : [0,1] \longrightarrow [0,1]$ **[14]**.

Problem 3. Is every Darboux function a superposition of (two) almost continuous functions?

5. Maximal additive subfamilies. Let Y be a family of real functions. A subfamily $\mathcal{M}_a(Y)$ of Y is called the maximal additive family for Y provided $\mathcal{M}_a(Y)$ is the set of all functions in Y such that $f+g \in Y$ whenever $f \in \mathcal{M}_a(Y)$ and $g \in Y$. Similarly we define the families:

$$\mathcal{M}_m(Y) = \{ f \in Y \mid \forall g \in Y \ \ f \cdot g \in Y \},$$

$$\mathcal{M}_{max}(Y) = \{ f \in Y \mid \forall g \in Y \ \ max(f,g) \in Y \},$$

$$\mathcal{M}_{min}(Y) = \{ f \in Y \mid \forall g \in Y \ \ min(f,g) \in Y \},$$

$$\mathcal{M}_l(Y) = \mathcal{M}_{max}(Y) \cap \mathcal{M}_{min}(Y).$$

Those families are well-known for the class of Darboux functions.

Theorem 4.

(1) $\mathcal{M}_a(\mathcal{D}) = \mathcal{M}_m(\mathcal{D})$ = *the family of all constant functions* **[15]**.

(2) $\mathcal{M}_{max}(\mathcal{D})$ *is the family of all upper semi-continuous functions with the Darboux property* **[6]**.

(3) $\mathcal{M}_{min}(\mathcal{D})$ *is the class of all lower semi-continuous functions with the Darboux property* **[6]**.

(4) $\mathcal{M}_l(\mathcal{D}) = \mathcal{C}$.

For the family of all almost continuous functions we have the following results **[9]**.

Theorem 5. *We have:*

(1) $\mathcal{M}_a(\mathcal{A}) = \mathcal{C}$,

(2) $\mathcal{M}_l(\mathcal{A}) = \mathcal{C}$,

(3) $\mathcal{M}_m(\mathcal{A}) = \{f : R \longrightarrow R \ / \text{if } x_0$ *is a point of right-hand (left-hand) discontinuity of* f *then* $f(x_0) = 0$ *and there exists a sequence* $((x_n)_n$ *such that* $x_n \searrow x_0$ $(x_n \nearrow x_0)$ *and* $f(x_n)=0$ *for* $n=1,2...\}$

Problem 4. Characterization of the classes $\mathcal{M}_{max}(\mathcal{A})$ and $\mathcal{M}_{min}(\mathcal{A})$.

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These notes collect some of the talks given in the Seminario del Departamento de Matemáticas Fundamentales de la U.N.E.D. in Madrid. Up to now the following titles have appeared:

1 **Luigi Grasselli,** Crystallizations and other manifold representations.
2 **Ricardo Piergallini**, Manifolds as branched covers of spheres.
3 **Gareth Jones**, Enumerating regular maps and hypermaps.
4 **J.C.Ferrando, M.López-Pellicer**, Barrelled spaces of class N and of class χ_0
5 **Pedro Morales**, Nuevos resultados en Teoria de la medida no conmutativa.
6 **Tomasz Natkaniec**, Algebraic structures generated by some families of real functions.
7 **Gonzalo Riera**, Algebras of Riemann matrices and the problem of units.
8 **Lynne D. James**, Representations of Maps.
9 **Grzegorz Gromadzki**, On supersoluble groups acting on Klein surfaces.
10 **Maria Teresa Lozano**, Flujos en 3-variedades.