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BARRELLED SPACES OF CLASS n AND OF CLASS X_0

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Abstract. We study the properties and the separation of the barrelled spaces of class n and χ_0 . We show that every non-normable Fréchet space contains a dense barrelled subspace of class $n-1$ which is not of class n . The totally barrelled spaces are barrelled spaces of class χ_0 and $l_0^\infty(X, \mathcal{A})$ is an example of a barrelled space of class χ_0 which is not totally barrelled.

From now onwards by "space" we mean "Hausdorff locally convex space over the field of the real or complex numbers". If A is a subset of a space $\langle A \rangle$ denotes its linear span. If B is a bounded absolutely convex subset of a space E , then E_B denotes the normed space over its linear hull. If $\{E_i, i \in I\}$ is a family of spaces, $E = \bigcap_{i \in I} E_i$ and J is a subset of I , then $E(J)$ denotes the subspace of E consisting of those elements of support J . A space E is Baire-like (unordered Baire-like), [9] ([10]), if given an increasing (arbitrary) sequence of closed absolutely convex subset of E covering E then one of them is a neighbourhood of the origin. E is suprabarrelled or (db), [13] ([11]), if given an increasing sequence of subspaces of E covering E then one of them is dense and barrelled. E is ordered suprabarrelled, [5], if given an increasing sequence of subspaces of E covering E there is one of them which is suprabarrelled. E is totally barrelled [15], if given a sequence of subspaces of E covering E there is one of them which is Baire-like.

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The suprabarrelled spaces are called in [8] barrelled spaces of class one. These spaces are Baire-like, and taking into account that every dense and barrelled subspace of a Baire-like space is always Baire-like ([1],16(5)) it results that E is suprabarrelled if given an increasing sequence of subspaces of E covering E there is one of them which is Baire-like.

Following [8], for each positive integer $n > 1$, E is called barrelled space of class n if given an increasing sequence $(E_n)_{n=1}^{\infty}$ of subspaces of E covering E there is an E_p which is barrelled of class $n-1$. We must note that since E is Baire-like we can choose the subspace E_p above being dense in E . E is called a barrelled space of class χ_0 if E is a barrelled space of class n , for $n=1,2,3,\dots$. The ordered suprabarrelled spaces are barrelled spaces of class 2 and in [6] is proved that $l_0^{\infty}(X, \mathcal{A})$ is a barrelled space of class χ_0 .

1. Permanence properties of the barrelled spaces of class n and of class χ_0 .

As it is well known, the barrelled spaces of class 1 and 2, and the barrelled and Baire-like spaces are stable under separated quotients, completions, countable-codimensional subspaces and arbitrary products, and satisfy the three-space problem ([10],[13],[3] and [5]). We are going to show that these properties are shared by each one of the barrelled spaces of class n . From these facts it follows immediately that the barrelled spaces of class χ_0 have these properties as well.

Proposition 1. *For every positive integer n , if E is a barrelled space of class n and F is a closed subspace of E , then E/F is a barrelled space of class n .*

Proof. For brevity, we will denote the barrelled spaces of class n by C_n . As the property is true for $n=1,2$, we will assume that it also holds for some n ,

and we will prove that it also holds for $n+1$. Let F be a closed subspace of $E \in C_{n+1}$. We must show that $E/F \in C_{n+1}$. If k is the canonical mapping from E onto E/F and $\{H_i\}_{i=1}^{\infty}$ is an increasing sequence of subspaces of E/F covering E/F there is some positive integer p such that $k^{-1}(H_p) \in C_n$. Then, by the induction hypothesis, as F is a closed subspace of $k^{-1}(H_p)$, we have that $H_p = k^{-1}(H_p)/F \in C_n$. Therefore $E/F \in C_{n+1}$.

Proposition 2. *For every positive integer n , if $F \in C_n$ and F is a dense subspace of a space E , then $E \in C_n$.*

Proof. The property is true for $n=1,2$. We assume that it holds for some $n \geq 2$, and we show that this property is also true for $n+1$. So let $F \in C_{n+1}$ be a dense subspace of some space E and let $\{E_i\}_{i=1}^{\infty}$ be an increasing sequence of subspaces of E covering E . Clearly, there exists a positive integer p such that $E_p \cap F$ is a dense subspace of F belonging to the class C_n . Since F is dense in E we have that $E_p \cap F$ is a dense subspace of E_p which belongs to C_n . The conclusion follows now from the induction hypothesis.

Proposition 3. *For every positive integer n , if $E \in C_n$ and F is a countable codimensional subspace of a space E , then $E \in C_n$.*

Proof. As the property is true for $n=1,2,\dots$, we are going to show that it holds for some $n+1$, when it is true for some $n \geq 2$. So let F be a countable codimensional subspace of $E \in C_n$ and let $\{F_i, i=1,2,\dots\}$ be an increasing sequence of subspaces of F covering F . If G denotes an algebraic complement of F in E , since $\{G+F_i, i=1,2,\dots\}$ is increasing and covers E , there is a positive integer p such that $G+F_p \in C_n$. The conclusion now raises from the induction hypothesis.

Proposition 4. For every positive integer n , if $\{E_i, i \in I\}$ is a family of spaces such that $E_i \in C_n$ for every $i \in I$, then the product $E = \prod_{i \in I} E_i \in C_n$.

Proof. We carry out the proof in three steps.

Step 1. If $I=\{1,2\}$ the property is true for $n=1,2$. We assume that property holds for some $n \geq 2$ and we show that it also holds for $n+1$. So we suppose that E_1 and E_2 are members of the class C_{n+1} and that $\{F_j\}_{j=1}^\infty$ is an increasing sequence of subspaces of E covering E . Clearly there is some $p \in \mathbb{N}$ such that $F_p \cap E_i$ is a dense subspace of E_i which belongs to C_n , and this for $i=1,2$. According to the induction hypothesis, $(F_p \cap E_1) \times (F_p \cap E_2)$ is a dense subspace of F_p belonging to C_n . The conclusion is now a consequence of proposition 2.

Step 2. If $I=\mathbb{N}$ the property is true for $n=1,2$. We assume that this property holds for some $n-1 \geq 2$ and that it does not hold for n . So we suppose that $E_i \in C_n$ for $i=1,2,\dots$ and that $E = \prod_{i \in \mathbb{N}} E_i \in C_{n-1} \setminus C_n$.

Therefore there is an increasing sequence $\{F_{j_1}\}_{j_1=1}^\infty$ of dense subspaces of E covering E such that $F_{j_1} \in C_{n-2} \setminus C_{n-1}$. Given j_1 there is an increasing sequence $\{F_{j_1 j_2}\}_{j_2=1}^\infty$ of dense subspaces of F_{j_1} covering F_{j_1} such that every $F_{j_1 j_2} \in C_{n-3} \setminus C_{n-2}$. Continuing in this way we obtain a countable family $F_{j_1 j_2 \dots j_{n-2}}$ of dense subspaces of E such that $F_{j_1 j_2 \dots j_{n-2}} \in C_1 \setminus C_2$. Given j_1, j_2, \dots, j_{n-2} we obtain an increasing sequence $\{F_{j_1 j_2 \dots j_{n-2} j_{n-1}}\}_{j_{n-1}=1}^\infty$ of dense subspaces of E which are Baire-like and not suprabarrelled and, therefore, given $j_1, j_2, \dots, j_{n-2}, j_{n-1}$ there is an increasing sequence $\{F_{j_1 j_2 \dots j_{n-2} j_{n-1} j_n}\}_{j_n=1}^\infty$ of dense subspaces of E which are not barrelled. Let $T_{j_1 j_2 \dots j_n}$ be a barrel of $F_{j_1 j_2 \dots j_n}$ which is not a neighborhood of the origin in $F_{j_1 j_2 \dots j_n}$, and let $B_{j_1 j_2 \dots j_n}$ denote its closure in E . Now we define

$$L_{j_1 j_2 \dots j_{n-1} j_n} := \langle B_{j_1 j_2 \dots j_n} \rangle, \quad G_{j_1 j_2 \dots j_{n-1} j_n} := \bigcap_{j \geq j_n} L_{j_1 j_2 \dots j_{n-1} j},$$

$$L_{j_1 j_2 \dots j_{n-1}} := \bigcup_{j=1}^{\infty} G_{j_1 j_2 \dots j_{n-1} j}$$

and so on until $L_{j_1} := \bigcup_{j=1}^{\infty} G_{j_1 j}$ and $G_{j_1} := \bigcap_{j \geq j_1} L_j$.

The sequences

$$\{G_{j_1}\}_{j=1}^{\infty} \text{ and } \{G_{j_1 j_2 \dots j_n}\}_{j=1}^{\infty}$$

are increasing and covers E and $L_{j_1 j_2 \dots j_k}$, being

$$G_{j_1 j_2 \dots j_k} \subseteq L_{j_1 j_2 \dots j_k} = \bigcup_{j=1}^{\infty} G_{j_1 j_2 \dots j_k j}$$

for $1 \leq k \leq n-1$.

There is some positive integer p such that G_p contains the subspace $E(p+1, p+2, \dots)$ of E . In fact, if this property were not true, then would exists some $x_r \in E(r+1, r+2, \dots) \setminus G_r$, $r=1, 2, \dots$. The projection of these points in every E_i are contained in a finite set, and therefore, $A = \overline{\{x_r, r=1, 2, \dots\}}$ is a Banach disk in E , and given that $\{G_s, s=1, 2, \dots\}$ covers E_A , there is some $s_1 \in \mathbb{N}$ such that for $s'_1 > s_1$ $G_{s'_1} \cap E_A$ is a Baire space which is a dense subspace of E_A , and, given $s'_1 > s_1$, as $\bigcup_{s_2=1}^{\infty} G_{s'_1 s_2}$ covers $G_{s'_1}$ there exists an $s_2(s'_1)$ such that for $s'_2 > s_2(s'_1)$, $G_{s'_1 s'_2}$ is a Baire space which is a dense subspace of E_A .

Continuing in this way we find $s_3(s'_1, s'_2), \dots, s_n(s'_1, s'_2, \dots, s'_{n-1})$ such that $G_{s'_1 s'_2 \dots s'_n} \cap E_A$ is a Baire space which is a dense subspace of E_A for $s'_1 > s_1, s'_2 > s_2(s'_1), \dots, s'_n > s_n(s'_1, s'_2, \dots, s'_{n-1})$.

Obviously $L_{s'_1 s'_2 \dots s'_n} \cap E_A$ is a Baire space which is a dense subspace of E_A , and the barrel $B_{s'_1 s'_2 \dots s'_n} \cap L_{s'_1 s'_2 \dots s'_n} \cap E_A$ is a zero-neighborhood in $L_{s'_1 s'_2 \dots s'_n} \cap E_A$, and from density, being $B_{s'_1 s'_2 \dots s'_n} \cap E_A$ closed in E_A , we have that $B_{s'_1 s'_2 \dots s'_n}$ contains λA , being $\lambda > 0$.

The inclusion $A \subseteq L_{s'_1 s'_2 \dots s'_n}$, for

$$s'_1 > s_1, \quad s'_2 > s_2(s'_1), \quad \dots, \quad s'_n > s_n(s'_1, s'_2, \dots, s'_{n-1}).$$

implies that $A \subseteq G_{s'_1 s'_2 \dots s'_n}$ if

$$s'_1 > s_1, \quad s'_2 > s_2(s'_1), \quad \dots, \quad s'_n > s_n(s'_1, s'_2, \dots, s'_{n-1}).$$

Obviously $A \subseteq L_{s'_1 s'_2 \dots s'_{n-1}}$ if

$$s'_1 > s_1, \quad s'_2 > s_2(s'_1), \quad \dots, \quad s'_{n-1} > s_{n-1}(s'_1, s'_2, \dots, s'_{n-2}),$$

and this inclusion implies that $A \subseteq G_{s'_1 s'_2 \dots s'_{n-1}}$ if

$$s'_1 > s_1, \quad s'_2 > s_2(s'_1), \quad \dots, \quad s'_{n-1} > s_{n-1}(s'_1, s'_2, \dots, s'_{n-2}).$$

Following these inclusions we obtain finally $A \subseteq G_{s'_1}$, for $s'_1 > s_1$, which contradicts that $x_{s'_1} \in A \setminus G_{s'_1}$.

On the other hand, since the sequence $\{G_s\}_{s=p}^\infty$ is increasing and covers the finite product $E(1,2,\dots,p)$, which belongs to the class C_n because of the step above, there is some $q_1 \geq p$ such that $G_{q_1} \cap E(1,2,\dots,p)$ is dense in $E(1,2,\dots,p)$ and belongs to the class C_{n-1} . This shows that G_{q_1} contains the subspace $\{G_{q_1} \cap E(1,2,\dots,p)\} \times E(p+1,p+2,\dots)$ which is dense in E and belongs to the class C_{n-1} as a consequence of the induction hypothesis and the step 1. By proposition 2 it follows that $G_{q_1} \in C_{n-1}$ and, a fortiori, $L_{q_1} \in C_{n-1}$. Thus there is some $q_2 \in \mathbb{N}$ such that $G_{q_1 q_2}$ is a dense subspace of E of class $n-2$. Following in this way we obtain $G_{q_1 q_2 \dots q_{n-1}}$ which is suprabarrelled and dense in E , and, finally $G_{q_1 q_2 \dots q_{n-1} q_n}$ which is dense in E and barrelled. This implies that $B_{q_1 q_2 \dots q_{n-1} q_n}$ is a neighborhood of the origin in E , which is a contradiction.

Step 3. We suppose now that I is non-countable and that $E_i \in C_n$ for some fixed positive integer n and for every $i \in I$. We denote as E_0 the subspace of E consisting of those vectors of E with countable support. Given that E_0 is

dense in E , according to proposition 2 we only need to prove that $E_0 \in C_n$. This is proved for $n=1$ in [13], theorem 5, and for $n=2$ in [5], theorem 3. We prove this property in general by induction. So we suppose that $E_i \in C_n$, for $i \in I$, and that $E_0 \in C_{n-1} \setminus C_n$.

Then, as in the step 2, changing E by E_0 , we obtain the countable family $\{F_{j_1 j_2 \dots j_n}, j_1, j_2, \dots, j_n = 1, 2, \dots\}$ of dense subspaces of E_0 which are not barrelled. Let $T_{j_1 j_2 \dots j_n}$ be a barrel of $F_{j_1 j_2 \dots j_n}$ which is not a neighborhood of the origin in $F_{j_1 j_2 \dots j_n}$, and let $B_{j_1 j_2 \dots j_n}$ denote its closure in E_0 .

We are going to show now that there is some $\langle B_{j_1 j_2 \dots j_n} \rangle$ which contains E_0 . On the contrary, let $x_{j_1 j_2 \dots j_n} \in E_0 \setminus \langle B_{j_1 j_2 \dots j_n} \rangle$ for $j_1, j_2, \dots, j_n = 1, 2, \dots$. Since every $x_{j_1 j_2 \dots j_n}$ has a countable support there exists some countable subset H of I such that $x_{j_1 j_2 \dots j_n} \in E(H)$, for $j_1, j_2, \dots, j_n = 1, 2, \dots$. By step 2 $E(H) \in C_n$, and, therefore, there exists n natural numbers p_1, p_2, \dots, p_n such that $F_{p_1 p_2 \dots p_n} \cap E(H)$ is a barrelled space which is dense in $E(H)$. Therefore, from density, $B_{p_1 p_2 \dots p_n}$ contains a neighborhood of zero in $E(H)$, which implies the contradiction $x_{p_1 p_2 \dots p_n} \in \langle B_{p_1 p_2 \dots p_n} \rangle$.

If $E_0 = \langle B_{p_1 p_2 \dots p_n} \rangle$, then the barrelledness of E_0 implies that $B_{p_1 p_2 \dots p_n}$ is a neighbourhood of zero in E_0 , and $T_{p_1 p_2 \dots p_n}$ is a zero neighbourhood in $F_{p_1 p_2 \dots p_n}$ which is a contradiction.

Proposition 5. *For every positive integer n , if E is a space and F is a closed subspace of E such that E/F and F belongs to the class C_n , then $E \in C_n$.*

Proof. Let k be the canonical mapping from E onto E/F . As for $n=1,2$ the proposition is proved in [13] and [6], we proceed by induction, assuming that this property holds for some $n-1 \geq 2$ and that it does not hold for n . So, if E/F and F belong to the class C_n and we suppose that $E \in C_{n-1} \setminus C_n$ then, proceeding

exactly as in proposition 4, step 2, we find a dense subspace $F_{j_1 j_2 \dots j_n}$ of E which is not barrelled and such that $F_{j_1 j_2 \dots j_n} \cap F$ and $k(F_{j_1 j_2 \dots j_n})$ are barrelled and dense in F and E/F respectively.

Let $T_{j_1 j_2 \dots j_n}$ be a barrel in $F_{j_1 j_2 \dots j_n}$ which is not a neighbourhood of the origin in $F_{j_1 j_2 \dots j_n}$, and let $B_{j_1 j_2 \dots j_n}$ denote its closure in E . The barrelledness of $F_{j_1 j_2 \dots j_n} \cap F$ determines the existence of a neighbourhood U of the origin in E such that $U \cap F_{j_1 j_2 \dots j_n}$ is contained in $B_{j_1 j_2 \dots j_n}$.

On the other hand, since $B_{j_1 j_2 \dots j_n} \cap U$ is absorbing in $F_{j_1 j_2 \dots j_n}$ and $k(F_{j_1 j_2 \dots j_n})$ is barrelled and dense in E/F , it follows that $k(B_{j_1 j_2 \dots j_n} \cap U)$ is a neighbourhood of zero in E/F . The contradiction follows from proposition 4 of [14].

2. Distinguishing between the different classes.

The chief aim of this section is to show that for each positive integer $n > 1$, every non-normable Fréchet space contains a dense subspace H which belongs to the class C_{n-1} but not to the class C_n .

Theorem 1. *For every positive integer $n > 1$ the product space ω contains a dense subspace $F \in C_{n-1} \setminus C_n$.*

Proof. This is true for $n=1,2$ (see[5] and [16]). Let us suppose that it is true for some $n \geq 2$ and let us show that it also holds for $n+1$. By hypothesis there is a dense subspace F of ω such that $F \in C_{n-1} \setminus C_n$. Using this subspace F we consider in $G = \omega^N$ the dense subspaces $E_r := \omega \times \dots \times \omega \times F \times F \times \dots$ and $E := \bigcup \{E_r : r=1,2,\dots\}$.

As $F \notin C_n$, then by proposition 1 $E_r \notin C_n$ and, therefore $E \notin C_{n+1}$.

Next we show that $E \in C_n$. On the contrary, we obviously have $E \in C_{n-1} \setminus C_n$,

and exactly as in the proposition 4, step 2, we determine in E the dense subspaces $F_{j_1 j_2 \dots j_n}$, $j_1, j_2, \dots, j_n = 1, 2, 3, \dots$, which are not barrelled, the barrel $T_{j_1 j_2 \dots j_n}$ of $F_{j_1 j_2 \dots j_n}$ which is not neighbourhood of the origin in $F_{j_1 j_2 \dots j_n}$, the closure $B_{j_1 j_2 \dots j_n}$ of $T_{j_1 j_2 \dots j_n}$ in E , and the subspaces

$$L_{j_1 j_2 \dots j_n}, G_{j_1 j_2 \dots j_n}, L_{j_1 j_2 \dots j_{n-1}}, \dots, L_{j_1} \text{ and } G_{j_1},$$

with $j_1, j_2, \dots, j_n = 1, 2, 3, \dots$. There is a positive integer p such that G_p contains $\{0\} \times \dots \times \{0\} \times F \times F \times \dots$. This fact follows as in proposition 4, step 2, or directly, considering in this proposition the product $E = \prod_{i \in \mathbb{N}} E_i$, with $E_i = F$, for $i = 1, 2, \dots$.

On the other hand, since the sequence $\{G_s\}_{s=p}^{\infty}$ is increasing and covers the Baire space ω^p , which obviously belongs to the class C_n , there is some $q_1 > p$ such that $G_{q_1} \cap \omega^p$ is a dense subspace of ω^p and belongs to the class C_{n-1} . This shows that G_{q_1} contains the subspace $\{G_{q_1} \cap \omega^p\} \times F \times F \dots$ which is dense in E and of the class C_{n-1} . By proposition 2 it follows that $G_{q_1} \in C_{n-1}$ and, a fortiori, $L_{q_1} \in C_{n-1}$. Thus, there is some $q_2 \in \mathbb{N}$ such that $G_{q_1 q_2}$ is a dense subspace of E of class $n-2$. Following in this way we obtain $G_{q_1 q_2 \dots q_{n-1}}$ which is suprabarrelled and dense in E , and, finally, $G_{q_1 q_2 \dots q_{n-1} q_n}$ which is dense in E and barrelled. This implies that $B_{q_1 q_2 \dots q_n}$ is a neighbourhood of the origin in E , and $T_{q_1 q_2 \dots q_n}$ a neighbourhood of the origin in $F_{q_1 q_2 \dots q_n}$, which is a contradiction.

Finally, as $G = \omega^{\mathbb{N}}$ is isomorphic to ω the conclusion follows.

Theorem 2. Every non-normable Fréchet space E contains a dense subspace F of the class C_{n-1} which is not of the class C_n .

Proof. We use here a standard argument based on the one hand in a known result of Eidelheit, [4], and in the other hand in proposition 5 above (see for

example [16]) for raising the desired conclusion.

3. Barrelled spaces of class χ_0 .

Proposition 6. *If E is a totally barrelled space, then E is a barrelled space of class χ_0 .*

Proof. If Γ stands for the class of all the totally barrelled spaces, it is obvious that Γ is contained in the class C_1 of the suprabarrelled spaces. We suppose that Γ is contained in C_n and we show now that Γ is contained in C_{n+1} . Actually, if $E \in \Gamma$, let $\{E_i, i=1,2,\dots\}$ an increasing sequence of subspaces of E covering E . As E is totally barrelled there exists some $p \in \mathbb{N}$ such that E_p is totally barrelled (see [15], theorem 4) and then, by the induction hypothesis, $E_p \in C_n$. This implies that $E \in C_{n+1}$.

The theorem 1 of [6] establishes that the space $l_0^\infty(X, \mathcal{A})$ is a barrelled space of class χ_0 . On the other hand, in [2] it is shown that this space is not totally barrelled. This provides the separation between the two classes. More examples of barrelled spaces of class χ_0 which are not totally barrelled may be obtained by the following proposition.

Proposition 7. *If E is a metrizable barrelled space of class χ_0 and F is an unordered Baire-like space, then $E \otimes_\mu F$ is a barrelled space of class χ_0 .*

Proof. If E is a metrizable space of class C_n , with $n \leq 2$, and F is unordered Baire-like, then $E \otimes_\mu F$ is also of the same class C_n (see, [12] and [5]). Therefore, if $E \otimes_\mu F$ were not barrelled space of class χ_0 there exists some positive integer $n > 2$ such that $E \otimes_\mu F$ does not belong to the class C_n , and, without loss of generality, we may suppose that $E \otimes_\mu F \in C_{n-1} \setminus C_n$. Now, as in

proposition 4, step 2, we obtain the countable family $\{M_{j_1 j_2 \dots j_n} : j_1, j_2, \dots, j_n = 1, 2, \dots\}$ of non barrelled dense subspaces of $E \otimes_{\mu} F$ covering $E \otimes_{\mu} F$ such that for every $s \in \{1, 2, \dots, n-1\}$ the sequence $\{M_{j_1 j_2 \dots j_{s+1}}\}_{j_{s+1}=1}^{\infty}$ is increasing and covers $M_{j_1 j_2 \dots j_s}$.

Let $T_{j_1 j_2 \dots j_n}$ be a barrel of $M_{j_1 j_2 \dots j_n}$ which is not a neighbourhood of the origin in $M_{j_1 j_2 \dots j_n}$ and let $B_{j_1 j_2 \dots j_n}$ be its closure in $E \otimes_{\mu} F$. Now if $\{U_r, r=1, 2, \dots\}$ is a decreasing base of closed absolutely convex neighbourhoods of zero in E , followings [12] we set $V_{j_1 j_2 \dots j_n r} := \{y \in F : U_r \otimes y \subseteq B_{j_1 j_2 \dots j_n}\}$. Clearly each $V_{j_1 j_2 \dots j_n r}$ is a closed absolutely convex subset of F . Since F is unordered Baire-like we only must prove that the countable family $\{V_{j_1 j_2 \dots j_n r} : j_1, \dots, j_n, r=1, 2, \dots\}$ covers the whole space F . In fact, if this were the case, some $V_{j_1 j_2 \dots j_n r}$ would be a neighbourhood of zero in F and, since $U_r \otimes V_{j_1 j_2 \dots j_n r}$ is contained in $B_{j_1 j_2 \dots j_n}$, $B_{j_1 j_2 \dots j_n}$ would be a neighbourhood of the origin in $E \otimes_{\mu} F$. Then $T_{j_1 j_2 \dots j_n}$ would be a neighbourhood of the origin in $M_{j_1 j_2 \dots j_n}$, which is a contradiction.

Now if y is any element of F we set $Y_{j_1 \dots j_s} := \{x \in E : x \otimes y \in M_{j_1 \dots j_s}\}$. It is clear that for every $s \in \{1, 2, \dots, n-1\}$ the sequence $\{Y_{j_1 \dots j_{s+1}}\}_{j_{s+1}=1}^{\infty}$ is increasing and covers $Y_{j_1 \dots j_s}$ and $\{Y_{j_1}\}_{j_1=1}^{\infty}$ is also increasing and covers the barrelled space of class $\chi_0 E$. Hence, there exists n positive integers p_1, p_2, \dots, p_n such that $Y_{p_1 p_2 \dots p_n}$ is dense in E and barrelled. Now, if we define $T := \{x \in E : x \otimes y \in B_{p_1 p_2 \dots p_n}\}$ it is clear that $T \cap Y_{p_1 p_2 \dots p_n}$ is a barrel in $Y_{p_1 p_2 \dots p_n}$, and therefore there is some neighbourhood of the origin U_r in E such that U_r is contained in T . Thus $U_r \otimes y \in B_{p_1 p_2 \dots p_n}$, that is to say, $y \in V_{p_1 p_2 \dots p_n r}$.

In [15] a totally barrelled normed space E such that $E \otimes_{\mu} l^2$ is not totally barrelled is obtained. By the proposition $E \otimes_{\mu} l^2$ is barrelled of class χ_0 .

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These notes collect some of the talks given in the Seminario del Departamento de Matemáticas Fundamentales de la U.N.E.D. in Madrid. Up to now the following titles have appeared:

- 1 **Luigi Grasselli**, Crystallizations and other manifold representations.
- 2 **Ricardo Piergallini**, Manifolds as branched covers of spheres.
- 3 **Gareth Jones**, Enumerating regular maps and hypermaps.
- 4 **J.C.Ferrando, M.López-Pellicer**, Barrelled spaces of class N and of class χ_0
- 5 **Pedro Morales**, Nuevos resultados en Teoria de la medida no conmutativa.
- 6 **Tomasz Natkaniec**, Algebraic structures generated by some families of real functions.
- 7 **Gonzalo Riera**, Algebras of Riemann matrices and the problem of units.
- 8 **Lynne D. James**, Representations of Maps.
- 9 **Grzegorz Gromadzki**, On supersoluble groups acting on Klein surfaces.
- 10 **Maria Teresa Lozano**, Flujos en 3-variedades.