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38

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AN ANALOGUE OF ROLLE'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

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Abstract

For "smooth" real functions $u(x, y)$ in a given domain, a simple inequality estimating lengths of level sets of u in terms of lengths of level sets of derivatives of u is obtained. The inequality establishes in fact "Rolle's theorem" for functions of two variable and also admits corresponding interpretations in complex analysis. On the other hand the inequality admits interpretations in hydrodynamics or electromagnetic fields and can be considered as a result in these subjects.

Let D be a domain with smooth boundary ∂D of finite Euclidean length $l(D)$ and let $u(x, y)$ be a function of two real variables which is continuously differentiable with its the first and second derivatives in the closure $\bar{D}$. Consider the zeros of $u(x, y)$, that is the set $l(u) := \{(x, y) \in \bar{D} \mid u(x, y) = 0\}$, usually called level sets of u, and assume, for simplicity, that the set does not degenerate so that $l(u)$ consists only on curves: here we consider intersection point of $l(u)$ (if there are some) as starting or terminal points of the curves. Thus we exclude a trivial case when u vanishes identically in some sub domains in $\bar{D}$. Denote by $c(\bar{D})$ the class of similar functions and by $L(D, f)$ the total length of all curves composing $l(f)$ for the function f.

We prove the following

Theorem 1. For any $u(x, y) \in c(\bar{D})$

$$L(D, u) \leq \sqrt{2}\left\{L(D, u'_x) + \sqrt{2}L(D, u'_y)\right\} + l(D). \tag{1}$$

This inequality can be regarded as a result in real analysis (as an analogue of Rolle's theorem for functions of two variables) and in complex analysis (Gamma-lines) as well as the inequality admit, likely different interpretations in the applied sciences (two interpretations are given below).

Inequality (1) as an analog of Rolle's theorem for real functions of two variables. At the early stage of mathematical education we learn that Rolle's theorem, that is between arbitrary two zeros of a real continuous function of one variable there is a zero of the derivative, is not true in general for complex functions. There are numerous attempts to obtain some analogous assertions for analytic functions w in a given domain D whose meaning is similar: for w satisfying certain hypothesis and having enough "powerful" set of zeros in a sub domain d the derivative w' also has zeros (some or all) in d. The following particular but astonishing Gauss' theorem was the first, and also till present the most simple and applicable, result of this kind: any disk involving all zeros of a given complex polynomial P involves also all zeros of derivative P'. Hundreds similar studies were made since then.

However, we are not aware whether someone tried to get an analogue of the Rolle's theorem in the real case, say for function $u(x, y) \in C^2$ of two real variables given in a domain D. In this case, we deal clearly with essentially different type of objects, since zeros of function u are in general curves but not points and also we have two derivatives u'_x and u'_y. But we still can look for a regularity whose meaning is essentially the same: if we have enough "powerful" set of zeros of u in D then u'_x and (or) u'_y also should have correspondingly "powerful" set of zeros. Now we can clearly see that inequality (1) is exactly this type of assertion and thus it can be considered as an analogue of the Rolle's theorem.

Interpretations of inequality (1) in terms of Gamma-lines in complex analysis. Let Γ be a curve in the complex plane, D be a domain in the complex plane and let w be a meromorphic function in $\bar{D}$. We define Gamma-lines (or better Gamma-set) of w in $\bar{D}$ as preimages $w^{-1}(\Gamma) \cap \bar{D}$. Note that the concept of Gamma-set is similar to that of the classical concept of a-points (that is set of $w^{-1}(a)$, $a \in \mathbb{C}$). So that we can follow classical studies in complex analysis related to these studies and consider Gamma sets in those numerous problems where previously a-points were studied. However it is pertinent to mention that investigations of Gamma-sets can bring a lot of new information both inside pure complex analysis in its applications in pure mathematics (see [1], [2]) and in applied topics. Particularly this is due to the circumstance that for the straight line $\Gamma := \{w | \operatorname{Im} w = h\}$ Gamma-lines are level sets of $\operatorname{Im} w$, which have a lot of interpretations in the applied science, also those mentioned above.

One of the natural concepts to be studied is the length $L(D, \Gamma, w)$ of Gamma-set.

At the end of 70s so called *tangent variation principle* has been established (see [1]) giving upper bounds for $L(D, \Gamma, w)$ for arbitrary w, D and large classes of Γ. It was applied to obtain for Gamma-sets some results analogous to the first and second fundamental theorems in Nevanlinna theory of a-points. Later on, upper bounds for Gamma-sets were studied for other particular classes of functions, domains and curves, see [1], p. 3.

Now we show that inequality (1) gives another type of upper bounds for $L(D, \Gamma, w)$.

Assume first that w is an analytic function in $\bar{D}$. Then making use of the following notations $X := \{w | \operatorname{Im} w = 0\}$, $Y := \{w | \operatorname{Re} w = 0\}$ and $w(z) = u(x, y) + iv(x, y)$ we note that $L(D, u) = L(D, X, w)$, $L(D, u'_x) = L(D, X, w')$ and $L(D, u'_y) = L(D, -v'_x) = L(D, Y, -w') = L(D, Y, w')$ so that inequality (1) yields

$$L(D, X, w) \leq \sqrt{2}\,\{L(D, X, w') + L(D, Y, w')\} + l(D). \tag{2}$$

The same is true for arbitrary meromorphic function w in $\bar{D}$. Indeed, we can remove from $\bar{D}$ small neighborhoods σ_i of zeros of w (finite number, since w is meromorphic function in the closure of D), then apply (2) in $\bar{D}\setminus \cup_i \sigma_i$, then let tend the diameters σ_i to zero. Since the lengths of $L(\sigma_i, X, w)$ tend to zero together with the diameter of σ_i (see [1], p. 20) it follows that (2) is valid also for our meromorphic function.

Inequality (2) can also be considered from another point of view. In Value distribution theory of meromorphic functions w in the complex plane we meet often results which compare numbers of zeros of w and numbers of zeros of w' in disks $\{z| \, |z| < r\}$ or compare Nevanlinna characteristic functions $T(r, w)$ and $T(r, w')$.

Clearly, inequality (2) is a similar assertion for Gamma-lines; fortunately it is valid for arbitrary domain.

Hydrodynamic and electromagnetic interpretations of inequality (1). Let $u(x, y)$ be an electric potential. Then solutions of $u(x, y) = A$ (equal to solutions of $u(x, y) - A = 0$) are equipotential lines, where the potential is equal to A. The magnitudes u'_x and u'_y mean components of the electric field in directions x and y; $u'_y = -v'_x$ since the corresponding complex potential w is an analytic function. Clearly, inequality (1) can now be read as a general theorem related to arbitrary electromagnetic field.

Another example. Let $u(x, y)$ be the velocity at the point (x, y) in a plane flow. Then solutions of $u(x, y) = A$ are those lines where the velocity is equal to A. The magnitudes u'_x and u'_y mean component of the accelerations in directions x and y; again $u'_y = -v'_x$. Thus all magnitudes in this inequality have hydrodynamic interpretations and inequality (1) can be considered as a theorem in hydrodynamics.

It is interesting that in both the above interpretations we can clearly see meaning of the inequality. Consider, what happen along a segment p with the endpoint (x_0, y_0) in the flow. Denote $u(x_0, y_0) = A$ and assume that the velocity u increases when we move along p in a small neighborhood of (x_0, y_0). This means that acceleration is positive in this neighborhood. Then to meet again this value A on p accelerations along p should necessarily decrease so that at some points on p this acceleration will be equal to zero. Inequality (1) gives a quantitative description of this physically clear phenomena.

An analogue of Theorem 1 for several variables and proofs will be given in another publication.

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