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ASPECTS OF THE GEOMETRY OF MAPPING CLASS GROUPS

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1 Introduction

The geometric structure of surface mapping class groups has been object of intense study in recent years, especially due to its connections with the geometry of Teichmüller space and the theory of hyperbolic 3-manifolds. Of particular importance in this sense is the proof of Thurston's Ending Lamination Conjecture, recently announced by Brock, Canary and Minsky [13] (see also [7] and [6] for another proof), building on work due mainly to Minsky. We remark that an alternative approach has been suggested by Rees [37].

We refer the reader to [26] for a thorough survey on surface mapping class groups. Let Σ be a surface of negative Euler characteristic, of genus g and p punctures (note that $3g - 3 + p \geq 0$). The *mapping class group* $\mathrm{Mod}(\Sigma)$ of Σ is the group of self-homeomorphisms of Σ up to homotopy. It is well-known that $\mathrm{Mod}(\Sigma)$ is finitely presented (see [23] and [41] for explicit presentations) and that it is generated by a finite collection of Dehn twists about simple closed curves on Σ. We want to study the group $\mathrm{Mod}(\Sigma)$ from the point of view of Geometric Group Theory, that is, we would like to examine actions of $\mathrm{Mod}(\Sigma)$ on different metric spaces (for which the mapping class group will act by isometries) in order to obtain information about $\mathrm{Mod}(\Sigma)$. This, in turn, will provide us with valuable information about the spaces that $\mathrm{Mod}(\Sigma)$ acts on.

In Section 2 we will describe three natural examples of spaces on which $\mathrm{Mod}(\Sigma)$ acts on: the Teichmüller space of Σ, complexes associated to Σ, such as the curve complex, and $\mathrm{Mod}(\Sigma)$ itself (or any Cayley graph for $\mathrm{Mod}(\Sigma)$).

In Section 3 we will introduce hyperbolic groups and explain why $\mathrm{Mod}(\Sigma)$ fails to be a hyperbolic group. In Section 4 we will discuss relatively hyperbolic group structures, and we will show that $\mathrm{Mod}(\Sigma)$ is not a "strong relatively hyperbolic group". Finally, in Section 5 we will present a generalisation of this result.

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2 Actions of $\mathrm{Mod}(\Sigma)$

2.1 Teichmüller space

We refer the reader to [24] for a detailed study of Teichmüller spaces. The Teichmüller space $T(\Sigma)$ of Σ is perhaps the most classical space on which $\mathrm{Mod}(\Sigma)$ acts. The space $T(\Sigma)$ is the set of finite area hyperbolic structures on $int(\Sigma)$ up to homotopy. More specifically, a point in $T(\Sigma)$ is an equivalence class $[(S, f)]$, where S is a finite area hyperbolic structure on $int(\Sigma)$ and $f : int(\Sigma) \to S$ is a homeomorphism, called the *marking* of S. Here, we declare two pairs (S, f) and (R, g) to be equivalent if and only if $g \circ f^{-1}$ is homotopic to an isometry between S and R. The mapping class group $\mathrm{Mod}(\Sigma)$ acts naturally on $T(\Sigma)$ by changing the marking.

Topologically, the space $T(\Sigma)$ is homeomorphic to $\mathbb{R}^{6g-6+2p}$. However, the geometry of $T(\Sigma)$ is much more complicated. The space $T(\Sigma)$ admits two classical metrics, for which the mapping class group is (except for some low-dimensional cases) the full isometry group by results due to Royden [38] and Masur-Wolf [32] respectively. They are:

1. *Teichmüller metric:* Equipped with this Finsler metric, $T(\Sigma)$ is a proper, geodesic metric space. We note that the Teichmüller metric on $T(\Sigma)$ is not non-positively curved in any sense (see [30] and [33]). However, Masur and Minsky [31] showed that the result of "coning-off" the thin regions of $T(\Sigma)$ is a Gromov-hyperbolic space (see Section 3 for definitions), but no longer locally compact.
2. *Weil-Petersson metric:* This is a Riemannian metric of negative sectional curvatures, although these are not bounded away from 0 (see, for instance, [42]). Endowed with this metric, $T(\Sigma)$ is a CAT(0) space (see [11] for definitions). However, the Weil-Petersson metric is not complete, the reason being that one can "pinch" geodesics on Σ in finite time. The completion of the Weil-Petersson metric is obtained by adding "noded" Riemann surfaces [29]. The resulting complete space is not locally compact anymore: a noded Riemann surfaces does not have a relatively compact neighbourhood in the completion. The action of $\mathrm{Mod}(\Sigma)$ extends a cocompact action on the completion, with quotient the Deligne-Mumford compactification of moduli space. Despite being CAT(0), the Weil-Petersson Teichmüller space is rarely negatively curved even in the weakest of senses: it is shown in [14] that the Weil-Petersson metric is Gromov-hyperbolic (see Section 3) if and only if the surface is a twice-punctured torus or a five-times punctured sphere. (See [2] for a somewhat simpler proof using only the Weil-Petersson geometry; see also [4] for another proof, which is similar in spirit to that in [14].) For more details on the Weil-Petersson metric we refer the reader to [42].

Remark 1 *In [8], Bowditch shows that if d is any complete Gromov hyperbolic metric on Teichmüller space, invariant under the action of the mapping class group, then the action of* $\mathrm{Mod}(\Sigma)$ *on Teichmüller space must be parabolic. Effectively, this implies that there is no natural metric of negative curvature on Teichmüller space, even in the weakest of senses. Thus we see that there is a strong obstruction on the geometry of Teichmüller space coming from the structure of the mapping class group.*

2.2 Complexes associated to surfaces

Another natural class of spaces on which $\mathrm{Mod}(\Sigma)$ acts are the so-called "complexes associated to hyperbolic surfaces". These are infinite, finite-dimensional connected CW-complexes built from topological information on Σ and for which $\mathrm{Mod}(\Sigma)$ is, except in a few cases, the full automorphism group. One source of difficulty when dealing with these complexes comes from the fact that they are normally locally infinite (each vertex has infinite valence). Among these complexes, the *curve complex* and the *pants complex* are of particular interest for us (let us remark that another complex, the *train-track complex* has been recently used by Hamenstädt to derive important properties for $\mathrm{Mod}(\Sigma)$). For the rest of the exposition, unless otherwise stated, by a *curve* on Σ we will mean a non-trivial free homotopy class of simple closed curves on Σ which are not homotopic to a puncture.

Definition 2 (Curve complex) *The curve complex $\mathcal{C}(\Sigma)$ is the simplicial complex whose vertex set is precisely the set of curves on Σ and where a collection A of curves spans a simplex if and only if we can represent the elements of A disjointly on Σ.*

The simplicial dimension of $\mathcal{C}(\Sigma)$ is clearly $3g - 4 + p$, that is, the number of curves in a pants decomposition of Σ minus 1. In the case of a three-holed sphere, or pair of pants, the curve complex is the empty set. For the four-holed sphere and the once-puncture torus, the curve complex is an infinite collection of isolated points; however, one can easily modify the definition to get a connected graph (the Farey graph see [31]).

We turn $\mathcal{C}(\Sigma)$ into a metric space by giving each simplex a Euclidean structure. The geometry of $\mathcal{C}(\Sigma)$ was studied by Masur and Minsky [31], where they showed that the curve complex is Gromov-hyperbolic (see Section 3). A somewhat simpler proof of this result is given in [9]. The curve complex was used by Harer [21] to study cohomological properties of mapping class groups; the automorphism group of $\mathcal{C}(\Sigma)$ was studied by Ivanov [25].

The pants complex $\mathcal{P}(\Sigma)$ is a 2-dimensional CW-complex whose vertices are pants decompositions of Σ and one connects two vertices by an edge if and only if the corresponding pants decompositions are related by an "elementary move" (see for instance [12] for definitions). One then fills certain simple loops of $\mathcal{P}(\Sigma)$, which correspond to algebraic relations in the mapping class group, with 2-cells. The pants complex has been successfully used to compute presentations for $\mathrm{Mod}(\Sigma)$ (see [23] and [41]).

Remark 3 *In addition, these two complexes encode the large-scale geometry (that is, up to quasiisometry, see Section 3) of Teichmüller space. More specifically, the curve complex $\mathcal{C}(\Sigma)$ is quasiisometric to the so-called "electric" Teichmüller space (see [31]) and the pants complex $\mathcal{P}(\Sigma)$ is quasiisometric to the Weil-Petersson metric (see [12]).*

2.3 Cayley graphs

In this subsection, we consider an arbitrary finitely generated group G. Following Gromov [20], we will consider the group G as a metric space, equipped with the *word metric* with respect to some generating set. The group G then acts by isometries on a metric space, itself, or equivalently it acts by isometries on the *Cayley graph* (see below) with respect to some generating set. From this apparently obvious consideration one can deduce an extraordinary number of properties the given group.

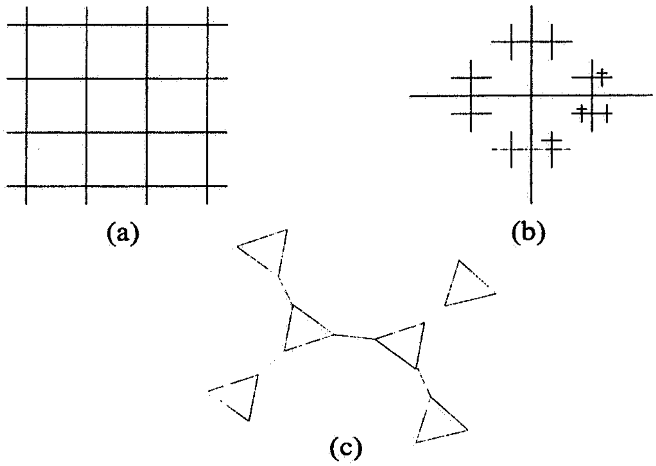

Figure 1: (a) $G = \mathbb{Z}^2$; (b) $G = \mathbf{F}^2$; (c) $G = \mathrm{PSL}(2, \mathbb{Z})$

Definition 4 (Cayley graph) *Let G be a finitely generated group and let Ω be a finite generating set (which one normally assumes to be symmetric). The Cayley graph* $\mathrm{Cay}(G, \Omega)$ *with respect to Ω is the graph whose vertices are the elements of G and where two vertices $g, g' \in G$ are joined by an edge if and only if $g' = \omega g$, for some $\omega \in \Omega$.*

Example 5 *Figure 1 shows (compact parts of) the Cayley graphs for $\mathbb{Z}^2$, $\mathbf{F}_2$ and* $\mathrm{PSL}(2, \mathbb{Z})$ *with respect to the standard symmetric generating sets.*

Note that words in the group G correspond to paths from 1_G in $\mathrm{Cay}(G, \Omega)$, and that relations in G correspond to closed paths in $\mathrm{Cay}(G, \Omega)$. We can turn any Cayley graph of a given group into a metric space by deeming each edge to have length 1 (note that this is essentially the word metric on G). The geometry of the group (or that of any of its Cayley graphs) has important algebraic consequences, for instance, if a group G is finitely generated and *hyperbolic* (see the next section) then it is finitely presented and satisfies a linear *isoperimetric inequality* [19].

In Section 3 will see that surface mapping class groups are never hyperbolic except for some low-dimensional cases. However, it will be possible to endow mapping class groups with a weak structure (but not a strong one) of relatively hyperbolic group, as we will discuss in Section 4.

3 Gromov-hyperbolicity

For a detailed discussion on Gromov hyperbolic spaces and groups see [19]. Let X be a metric space, which from now on we assume to be a geodesic space, that is, every two points of X can be joined by a path in X of length equal to the distance between them. This assumption on X, despite being quite restrictive, simplifies the exposition in great manner; for a discussion in a more general setup, see [19]. By a triangle in X we mean three points $x, y, z \in X$ and three geodesic paths connecting them pairwise. Given a set $T \subseteq X$ and $r \geq 0$, we will denote by $N(T, r)$ the neighbourhood of T of radius r, that is, the set $\{x \in X \mid d(x, T) \leq r\}$, where d represents the distance function on X.

Definition 6 (Thin triangles) *Let X be a geodesic metric space and let Δ be a triangle in X with sides a_1, a_2, a_3. Let $\delta \geq 0$. We say that Δ is δ-thin if $a_i \subseteq N(a_j \cup a_k, \delta)$, for all $i, j, k \in \{1, 2, 3\}$ distinct.*

Definition 7 *Let X be a geodesic metric space. We say that X is δ-hyperbolic if there exists $\delta \geq 0$ such that every triangle in X is δ-thin. If the value of the constant δ is not relevant we will simply say that X is Gromov hyperbolic.*

Example 8 *It is an easy calculation to see that hyperbolic plane $\mathbb{H}^2$ is Gromov hyperbolic (in fact, this is true for $\mathbb{H}^n$, for all $n \in \mathbb{N}$). Any complete, simply-connected manifold of pinched negative sectional curvatures is Gromov hyperbolic (see [19], p.52). A less expectable example of a Gromov hyperbolic space is given by a (simplicial) tree, that is, a graph with no loops: these spaces are 0-hyperbolic. On the other hand, Euclidean plane $\mathbb{E}^2$ is obviously not Gromov hyperbolic, and the same happens for any space that contains an isometric copy of $\mathbb{E}^2$.*

A much more surprising example of a Gromov hyperbolic space is the curve complex of a surface, as we mentioned in Section 1.2. This is the content of the following remarkable result, due to Masur and Minsky [31].

Theorem 9 (Hyperbolicity of the curve complex, [31]) *Let Σ be a surface of negative Euler characteristic and consider the curve complex $C(\Sigma)$ of Σ. If $C(\Sigma)$ is connected, then it is Gromov hyperbolic.*

The concept of Gromov hyperbolicity is very robust, as it is invariant under quasi-isometries, that is, maps that preserve distances up to a bounded additive constant. Roughly speaking, two spaces are quasiisometric if they look the same when looked at from far away. Formally, we have the following definition.

Definition 10 (Quasi-isometries) *Let (X_1, d_1) and (X_2, d_2) be geodesic metric spaces. We say that a map $\phi : X_1 \to X_2$ is a (λ, r, c)-quasiisometry if there are numbers $\lambda \geq 1, r \geq 0, c \geq 0$ such that the following is satisfied:*

1. $\frac{1}{\lambda} d_1(x, y) - r \leq d_2(\phi(x), \phi(y)) \leq \lambda d_1(x, y) + r$, *for all $x, y \in X_1$ and,*
2. *For all $z \in X_2$ there is $x \in X_1$ such that $d_2(z, \phi(x)) \leq c$.*

We say that X_1 and X_2 are quasiisometric if there is a (λ, r, c)-quasiisometry between them, for some $\lambda \geq 1, r \geq 0, c \geq 0$.

Example 11 *A metric space of bounded diameter is quasiisometric to a point. The lattice $\mathbb{Z}^n$ is quasiisometric to Euclidean space $\mathbb{E}^n$ of the same dimension, and $\mathbb{Z}^n$ is quasiisometric to $\mathbb{Z}^m$ if and only if $m = n$. The Cayley graph of $\mathrm{PSL}(2, \mathbb{Z})$ (see Fig. 1 (c)) is quasi-isometric to the simplicial 3-valent tree. The Farey graph is quasi-isometric to a simplicial tree (a proof of this fact can be found in [3] and it is due, in that form, to Brian Bowditch)*

The next result states the invariance of Gromov hyperbolicity under quasiisometries. For a proof, see [19].

Proposition 12 *Let X_1 and X_2 be geodesic metric spaces and let $\phi : X_1 \to X_2$ be a quasiisometry between them. Then, X_1 is Gromov hyperbolic if and only if X_2 is Gromov hyperbolic (although the hyperbolicity constants will not coincide, in general).*

The following result is an easy exercise on finitely generated groups (see [19]) but it will be crucial in the definition of hyperbolic group.

Lemma 13 *Let G be a finitely generated group and let Ω_1, Ω_2 be finite generating sets. Then, the Cayley graphs $\mathrm{Cay}(G, \Omega_1)$ and $\mathrm{Cay}(G, \Omega_2)$ are quasiisometric.*

In light of this observation, if $\mathrm{Cay}(G, \Omega)$ is Gromov hyperbolic then any Cayley graph for G is Gromov hyperbolic, by Proposition 12. We are finally ready to give a definition of hyperbolicity for finitely generated groups. We remark that there are several (equivalent) definitions of hyperbolicity for groups, see [19].

Definition 14 (Hyperbolic groups) *We say that a finitely generated group is hyperbolic if $\mathrm{Cay}(G, \Omega)$ is Gromov hyperbolic for some (and hence, any) finite generating set Ω.*

Example 15

1. *The free group $\mathbf{F_n}$ on n generators is hyperbolic, for all n.*
2. $\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ *is hyperbolic, since its classical Cayley graph (see Fig. 1) is quasiisometric to a tree.*
3. *The fundamental group $\pi_1(M)$ of a compact manifold M of pinched negative sectional curvatures is a hyperbolic group, since it is quasiisometric to the hyperbolic space of the same dimension as M (see [19], p.62).*
4. *On the other hand, $\mathbb{Z}^n$ is not hyperbolic for all $n \geq 2$.*
5. *In fact, any group containing a free abelian group of rank ≥ 2 cannot be hyperbolic. Thus* $\mathrm{Mod}(\Sigma)$ *is not hyperbolic unless the surface is a once-punctured torus or a four-times punctured sphere, in which case* $\mathrm{Mod}(\Sigma) \cong \mathrm{PSL}(2, \mathbb{Z})$.

4 Relatively hyperbolic structures for mapping class groups

We have just seen that, with the exception of some low-dimensional cases, surface mapping class groups are never hyperbolic since they contain high-rank abelian subgroups. However, one can still ask whether mapping class groups are "hyperbolic relative to" a collection of subgroups. In fact, Masur and Minsky [31] showed the following result which is to be regarded as a corollary to Theorem 16.

Theorem 16 (Weak relative hyperbolicity, [31]) *Let Σ be a hyperbolic surface of genus g and p punctures, with $3g-3+p \geq 2$, and consider its mapping class group* $\mathrm{Mod}(\Sigma)$. *Let $c_1, \ldots, c_n$ a transversal for the action of* $\mathrm{Mod}(\Sigma)$ *on the set of curves on Σ. Then* $\mathrm{Mod}(\Sigma)$ *is weakly hyperbolic relative to $H(c_1), \ldots, H(c_n)$, where $H(c_i)$ is the stabiliser of the curve c_i in* $\mathrm{Mod}(\Sigma)$, *for all $i = 1, \ldots, n$.*

Relatively hyperbolic groups, introduced by Gromov [20] and developed mainly by Farb [17], Szczepański [39] and Bowditch [10], provide a natural generalisation of both hyperbolic groups and geometrically finite kleinian groups. Let us note that there are two existing notions of relative hyperbolicity: one due to Farb [17], which we will refer to as "weak relative hyperbolicity"; and another one, due to Gromov-Szczepański-Bowditch, which we will refer to as "strong relative hyperbolicity", for reasons that will be apparent shortly. Primary examples of strong relatively hyperbolic groups are fundamental groups of finite-volume manifolds of pinched negative sectional curvatures (compare with Example 15.3).

Let us begin with the definition of weak relative hyperbolicity. Let G be a (finitely generated) group and let $H_1, \ldots, H_n$ be a finite collection of proper subgroups of G. Consider a finite generating set Ω for G and let $\mathrm{Cay}(G, \Omega)$ be the Cayley graph of G with respect to Ω. We form an augmentation $\mathrm{Cay}^*(G, \Omega)$ of $\mathrm{Cay}(G, \Omega)$ as follows: for each translate gH_i of H_i we add a new vertex $v(gH_i)$, which we connect to each vertex of gH_i by an edge of length $1/2$ (so that the region gH_i has diameter 1 in $\mathrm{Cay}^*(G, \Omega)$). Following Farb [17] we will refer to $\mathrm{Cay}^*(G, \Omega)$ as the "coned-off Cayley graph" of G with respect to Ω.

Definition 17 *With the same notation as above, we say that G is weakly hyperbolic relative to the subgroups $H_1, \ldots, H_n$ if the coned-off Cayley graph* $\mathrm{Cay}^*(G, \Omega)$, *with respect to some generating set Ω, is Gromov hyperbolic.*

Remark 18 *Farb shows [17] that if the coned-off Cayley graph with respect to some generating set is Gromov hyperbolic then the same holds for any generating set. Therefore, the notion of a weak relatively hyperbolic group is independent of a generating set.*

We now give the definition of strong relative hyperbolicity. In his paper [10], Bowditch gives two equivalent definitions of (strong) relatively hyperbolicity, of which we recall the second. We note that a graph $\mathcal{L}$ is said to be *fine* if for any vertices v, v' of $\mathcal{L}$ and any $n \in \mathbb{N}$ there are only finitely many injective paths from v to v' of length n.

Definition 19 (Strong relative hyperbolicity) *Let G be a finitely generated group and $H_1, \ldots, H_n$ a collection of proper finitely generated subgroups of G. We say that G is strongly hyperbolic relative to $H_1, \ldots, H_n$ if G admits an action on a connected graph $\mathcal{K}$ with the following properties:*

1. *$\mathcal{K}$ is fine and Gromov hyperbolic,*
2. *There are only finitely many G-orbits of edges and the stabiliser of every edge is finite,*
3. *The conjugates of $H_1, \ldots, H_n$ are precisely the stabilisers of the vertices of $\mathcal{K}$ of infinite valence.*

In his paper [17], Farb shows that many important examples of weak relatively hyperbolic groups satisfy a further property which he calls "Bounded Coset Penetration", or BCP for short. This is a geometric condition on fellow-travelling geodesic paths in $\mathrm{Cay}(G, \Omega)$ that enter cosets of $H_1, \ldots, H_n$. It turns out that weak relative hyperbolicity plus BCP is equivalent to strong relative hyperbolicity [16]. However, the BCP property is crucial: there are weak relative hyperbolic groups that are not strong relatively hyperbolic groups. For instance, the group $\mathbb{Z}^2$ is weakly, but not strongly, hyperbolic relative to the subgroup $\{(m, m) \mid m \in \mathbb{N}\}$. As we will shortly see, mapping class groups constitute another example of this behaviour.

The notion of relative hyperbolicity has important algebraic consequences (see [17] and [10]). In particular, there is a certain "tranference property" in strong relative hyperbolic groups, as it is often possible to deduce that the group G has a given property provided that the subgroups $H_1, \ldots, H_n$ have the property. Examples are *uniform embeddability in Hilbert space* [15], *exactness* [36] and *finite asymptotic dimension* [34]. In light of this, identifying a strong relatively hyperbolic group structure for a given group becomes an interesting problem. However, for the case of mapping class groups, we have the following result.

Theorem 20 ([1]) *Let Σ be a hyperbolic surface of genus g and p punctures, with $3g - 3 + p \geq 2$. Then, there is no finite collection $H_1, \ldots, H_n$ of proper subgroups of $\mathrm{Mod}(\Sigma)$ such that $\mathrm{Mod}(\Sigma)$ is strongly hyperbolic relative to $H_1, \ldots, H_n$.*

Remark 21 *It was already known (see, for instance, [10]) that $\mathrm{Mod}(\Sigma)$ cannot be strongly hyperbolic relative to the subgroups described in Theorem 16. Indirect proofs of Theorem 20 can be obtained from [8] and in [28]. Here, as well as in [1], we will give a hands-on proof of the result. Behrstock, Druţu and Mosher [5] have recently found an alternative proof of Theorem 20 using the asymptotic cone of the mapping class group.*

The main ingredients for our proof of Theorem 20 will be the next two Lemmas on relatively hyperbolic groups. We note that these results are well-known: the first result follows from work of Tukia [40], the second result is implicit in the work of Farb [17] and Bowditch [10] and is explicitly stated in [35].

Lemma 22 *Let G be a finitely generated group and let $H_1, \ldots, H_n$ be a finite collection of proper subgroups of G. Suppose that G is strongly hyperbolic relative to $H_1, \ldots, H_n$. If A is an abelian subgroup of G of rank at least two, then A is contained in some H_j, up to conjugation.*

Lemma 23 *Let G be a finitely generated group and let $H_1, \ldots, H_n$ be a finite collection of proper subgroups of G. Suppose that G is strongly hyperbolic relative to $H_1, \ldots, H_n$. Then,*

1. *For any $g_1, g_2 \in G$, the intersection $g_1 H_j g_1^{-1} \cap g_2 H_k g_2^{-1}$ is finite for $1 \leq j \neq k \leq n$.*
2. *For $1 \leq j \leq n$, the intersection $H_j \cap g H_j g^{-1}$ is finite for any $g \notin L_j$.*

We are finally ready to give a proof of Theorem 20.

Proof [of Theorem 20] Suppose, for contradiction, that $\mathrm{Mod}(\Sigma)$ were strongly hyperbolic relative to the proper subgroups $H_1, \ldots, H_n$.

First, we claim that H_i cannot contain a (power of a) Dehn twist. For, suppose that there exists a curve c_0 on Σ and an integer $n_0 \geq 1$ so that $\tau_0^{n_0} \in H_j$, where τ_0 is the (right) Dehn twist about c_0. Since $3g - 3 + p \geq 2$, by hypothesis, the surface Σ cannot be a pair of pants, a four-holed sphere nor a once-punctured torus. Therefore there exists a curve c_1 on Σ such that c_0 and c_1 are disjoint. Consider the Dehn twist τ_1 about c_1. Then τ_0 and τ_1 commute since c_0 and c_1 are disjoint. In particular, we must have that $\langle \tau_0^{n_0} \rangle \subset H_j \cap \tau_1 H_j \tau_1^{-1}$. However, since $\langle \tau_0^{n_0} \rangle$ is infinite, Lemma 23 implies that $\tau_1 \in H_j$. Now take any Dehn twist $\tau \in \mathrm{Mod}(\Sigma)$ about a curve c on Σ. Since the curve complex is connected (recall that we are assuming that $3g - 3 + p \geq 2$), there is a sequence $c_0, c_1, \ldots, c_m = c$ of simple curves so that c_{k-1} and c_k are disjoint for $1 \leq k \leq m$. By the argument we have just used we get that if $\tau_{k-1} \in H_j$, then $\tau_k \in H_j$, for all $k = 1, \ldots, m$ and $j = 1, \ldots, n$. In particular, we have that the Dehn twist τ about c belongs to H_j. But this is impossible, since we know that $\mathrm{Mod}(\Sigma)$ is generated by (finitely many) Dehn twists and we are assuming that the subgroups $H_1, \ldots, H_n$ are proper. Thus the claim follows.

Consider now two Dehn twists $\tau, \tau' \in \mathrm{Mod}(\Sigma)$ about disjoint curves c, c' on Σ. Since τ, τ' commute and are distinct, they generate a rank 2 abelian subgroup A of $\mathrm{Mod}(\Sigma)$. Thus, by Lemma 22, we see that A is conjugate into some H_j. However, this contradicts the claim in the previous paragraph. **QED**

5 Generalisations

In fact, Theorem 20 and its proof are just part of a much more general result, applicable to a wide collection groups, see [1]. Let us begin with a definition.

Definition 24 (Commutativity graph, [1]) *Let G be a group and let Ω be a (possibly infinite) generating set for G, all of whose elements have infinite order. The* commutativity graph $K(G, \Omega)$ *for G with respect to Ω is the simplicial graph whose vertex set is Ω and in which distinct vertices $s, s' \in \Omega$ are connected by an edge if and only if there are non-zero integers n_s, $n_{s'}$ so that $\langle s^{n_s}, (s')^{n_{s'}} \rangle$ is abelian.*

Abusing notation, we are making no difference between the elements of Ω and the vertices of $K(G, \Omega)$. We have the following result, see [1].

Theorem 25 (Non-relative hyperbolicity, [1]) *Let G be a finitely generated group. Suppose there exists a (possibly infinite) generating set Ω of cardinality at least two such that every element of Ω has infinite order and $K(G, \Omega)$ is connected. Suppose further that there exist adjacent vertices s, s' of $K(G, \Omega)$ and non-zero integers n_s, $n_{s'}$ so that $\langle s^{n_s}, (s')^{n_{s'}} \rangle$ is rank 2 abelian. Then, G is not strongly hyperbolic relative to any finite collection of proper finitely generated subgroups.*

This result represents a generalisation of Theorem 20. In the case of mapping class group, we took the set of all Dehn twists as the generating set Ω, for which the commutativity graph is the 1-skeleton of curve complex, which is connected. The proof of Theorem 25 is totally analogous to the proof of Theorem 20, just replacing the set of Dehn twists and the curve complex by an arbitrary generating set Ω and the commutativity graph $K(G,\Omega)$, respectively. Besides surface mapping class groups, examples of groups for which Theorem 25 applies: the Torelli group $\mathcal{I}(\Sigma)$ of a compact hyperbolic surface of genus $g \geq 3$ (see [27] for definitions); the special automorphism group $\mathrm{Aut}^{+}(\mathbf{F}_n)$ and the special outer automorphism group $\mathrm{Out}^{+}(\mathbf{F}_n)$ of the free group $\mathbf{F}_n$ (see [18] for definitions), $SL(n,\mathbb{Z})$ for $n \geq 5$; the 3-dimensional Heisenberg group, and Thompson's group F.

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