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THE SCHWARZ-CHRISTOFFEL CONFORMAL MAPPING FOR "POLYGONS" WITH INFINITELY MANY SIDES

The Schwarz-Christoffel conformal mapping for "polygons" with infinitely many sides

G. Riera, H. Carrasco, R. Preiss

This paper is devoted to the extension of the classical Schwarz-Christoffel transformation of the upper-half plane to the cases where the polygons need not be finite. We explore two possible generalizations, to polygons with infinite sides, as in a stair, and to fractals.

The mappings are in each case defined by an explicit formula and we explain how to generate the images by a standard mathematical software.

Many questions remain open and we point to some of them.

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Introduction

Let D be a polygon in the complex plane with n vertices and interior angles $\pi\,\alpha_i$, $0 \leq \alpha_i \leq 2$, $1 \leq i \leq n$; the exterior angles are given by $\pi\mu_i$ if $\alpha_i + \mu_i = 1$.

The Schwarz-Christoffel conformal mapping from the upper-half plane onto D is given by the formula

$$f(z) = \alpha \int_0^z \frac{dx}{(x-a_i)^{\mu_1}\cdots(x-a_n)^{\mu_n}} + \beta \tag{1}$$

for real numbers $a_1 < \cdots < a_n$ and constants α and β.

There is ample literature on the subject, see for example [1], [3], [2]. We find it necessary however to make some remarks on this formula.

First, even though (1) is explicit enough there is no known relation between the values of the a_i 's and the lengths of the sides of the polygon. This implies that we may not take an infinite product in the integrand without any justification; yet this is the kind of formulas we consider here. Also, the formula (1) is a necessary but not a sufficient one, since for some values of the a_i 's the mapping is not one to one (see [4]). To establish the necessity one needs to consider the behavior of $f(z)$ at ∞, a possibility also clearly excluded for infinite points a_i.

We will also need some variations of (1) which we recall.

One of the points a_i may be located at ∞, the formula remaining the same, and one of the vertices of the polygon may be at ∞, where we use the relation $\mu_1 + \cdots + \mu_n = 2$.

The formula for mapping the unit disc conformally onto D is equally the same one, where now $|a_i| = 1$, $1 \leq i \leq n$ and the formula for the conformal mapping onto the exterior of the polygon is given by

$$g(z) = \alpha \int_1^z e^{\lambda x}(x-a_1)^{\mu_1}\cdots(x-a_n)^{\mu_n}\,\frac{dx}{x^2} + \beta$$

with $\lambda = \sum_{i=1}^{n} \mu_i \,/\, a_i$

This paper explores the possibility of infinitely many points a_i in the real line (or the unit circle). We obtain results for two kinds of "polygons".

a) Polygons with an infinite number of sides. This is the case for example of an infinite stair with internal angles alternatively equal to $\pi/2$ or $3\pi/2$. The formula in this case is

$$f(z) = \int_0^z \sqrt{tan(x)}\,dx$$

The zeros of the denominator $cos(x)$ correspond to $a_i = (2i+1)\pi/2$, $\mu_i = 1/2$ and the zeros of the numerator sin (x) to $a_i = i\pi, \mu_i = -1/2$.

b) Fractals. This is the case for example of the interior of Koch's snowflake where the formula on the interior of the unit disc is

$$g(z) = \int_0^z \prod_{n=0}^{+\infty} (1 + x^{6\cdot 4^n})\,dx$$

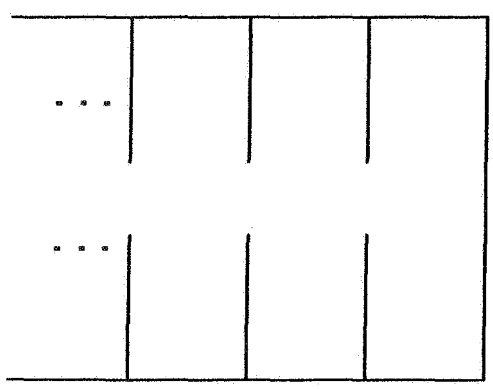
A similar generalization of (2) gives the conformal mapping onto the exterior of Koch's snowflake.

To obtain the figures in the text the reader will need a mathematical software such as Derive, Maple or Mathematica; we explain in each case how to obtain them.

As far as we know none of these formulas have been considered before and this same novelty of the subject precludes complete and general results. We may hope that this paper will motivate the reader to look into more diverse results and perhaps general theorems.

We point for example at an open problem:

Find the formula for a conformal mapping of the upper-half plane into the interior of a polygon of the following kind



(the formula without the two angles $\pi/2$ is obtained here). This mapping is of importance in the study of differentials on hyperelliptic surfaces of infinite genus as considered in a pioneering work by Myrberg, see [5].

Polygons with an infinite number of sides

(1) **The infinite stair.**

The first example we consider is the mapping defined on the upper half plane by the formula

$$f(z) = \int_0^z \sqrt{tan(x)}\, dx$$

The image is given in Figure 1 where the curves are the images of horizontal and vertical segments; they can be defined directly by

$$\begin{aligned} \alpha(s) &= (Re\ (Numint(\sqrt{tan(x)},\ x = 0 \text{ to } 0.6 + si)), \\ &\quad Im\ (Numint\ (\sqrt{tan\,(x)}, x = 0 \text{ to } 0.6 + si))) \\ &\quad 0.1 \le s \le 1.1 \end{aligned}$$

for a vertical segment and

$$\begin{aligned} \beta(s) &= (Re\ (Numint\ (\sqrt{tan\,(x)},\ x = 0\ to\ 0.09i + s)), \\ &\quad IM\ (Numint\ (\sqrt{tan(x)},\ x = 0\ to\ 0.09i + s))) \\ &\quad 0.4 \le s \le 10 \end{aligned}$$

for a horizontal segment.

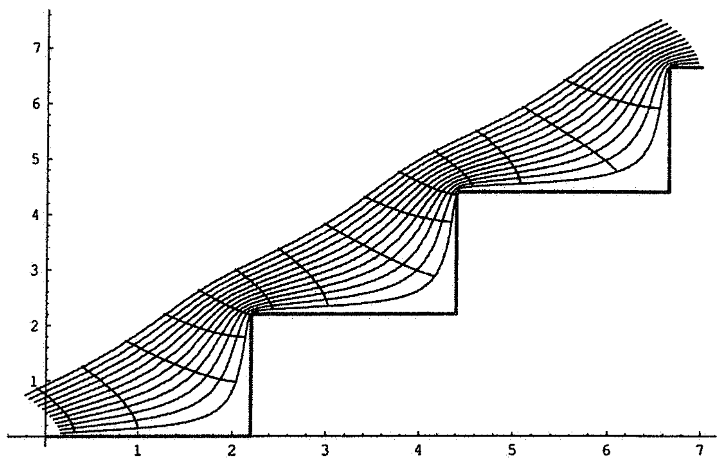


FIGURE 1. The infinite stair. $f(z) = \int_0^z \sqrt{tan\,x}\, dx$

The image of the real axis is the infinite stair with steps of equal length

$$\int_0^{\pi/2} \sqrt{tan(x)}\, dx = \pi\sqrt{2}/2\,.$$

To actually prove that this formula defines a conformal mapping we proceed as follows.

Consider the triangle of Figure 2.

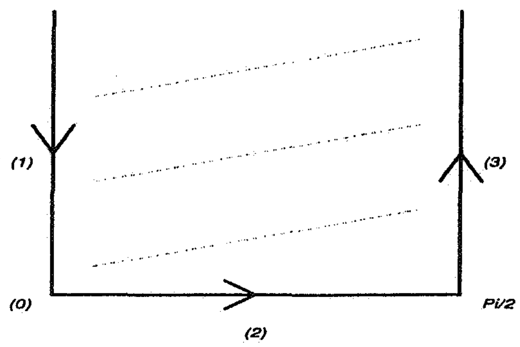


FIGURE 2.

The image of half the imaginary axis (1) is given by

$$\int_0^{is} \sqrt{tan(x)}\, dx = i\int_0^{s} \sqrt{tan(it)}\, dt = \frac{-1+i}{\sqrt{2}} \int_0^{s} \sqrt{\frac{e^t - e^{-t}}{e^t + e^{-t}}}\, dt$$

for $0 \leq s \leq +\infty$. Thus it is the half line making an angle of $3\pi/4$ with the positively oriented real axis. The image of (2) is the real segment $[0, \pi\sqrt{2}/2]$ and the image of (3) is a half line parallel to the first one beginning at $\pi\sqrt{2}/2$ as in Figure 3.

Consider the triangle of Figure 3.

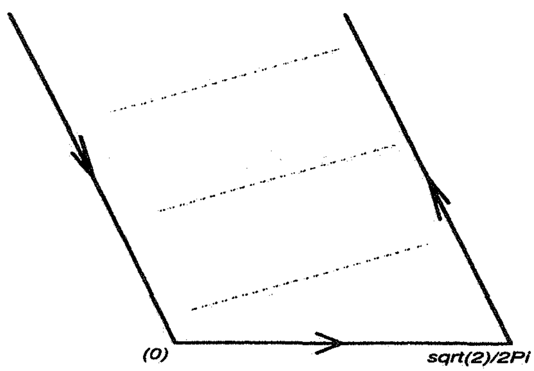


FIGURE 3.

The general theory of conformal mappings applies to these two triangles so that from the bijection on the boundaries we conclude conformality in the interior. Using now Schwartz's reflection principle as applied to the vertical segments (1), (3) and their reflections the mapping can be extended to the entire upper half-plane.

This formula can be generalized to zig-zag patterns via

$$f(z) = \int_0^z \frac{sin^\nu(x)}{cos^\mu(x)}\, dx$$

$0 < \nu,\ \mu < 1)$

(2) The hairy half-plane.

We consider now interior angles $\pi/2$ at $a_i = (2i+1)\pi/2)$ $(i \in \mathbb{Z})$ and 2π $atb_i = (2i+1)\pi$.

The formula is therefore

$$f(z) = \int_0^z \frac{cos(x/2)}{\sqrt{cos(x)}}\, dx$$

and the image is in Figure 4.

To establish conformality it is first necessary to prove that the length of the segment from an angle $\pi/2$ to the point of the hair is equal on both sides, i.e. that

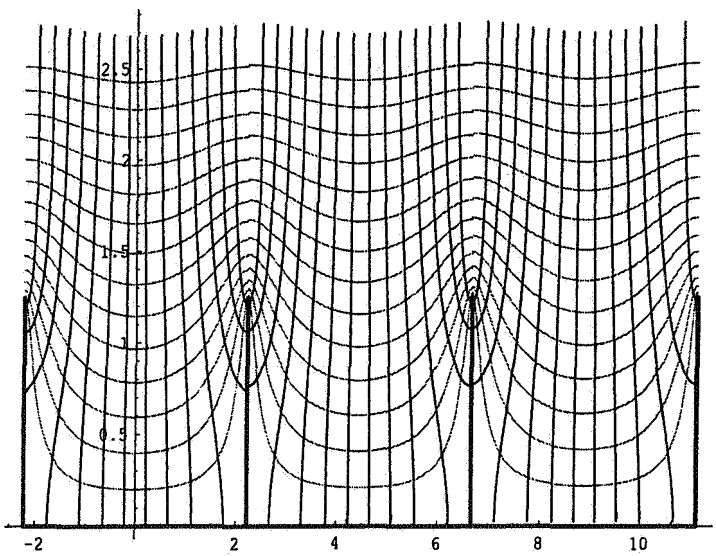


FIGURE 4. The hairy half-plane. $f(z) = \int_0^z \frac{cos(x/2)}{\sqrt{cos\,(x)}}\, dx$

$$\int_{\pi/2}^{\pi} \frac{cos(x/2)}{\sqrt{-cos(x)}}\,dx = \int_{\pi}^{3\pi/2} \frac{-cos(x/2)}{\sqrt{-cos(x)}}\,,$$

an equality easily obtained by the change of variable $x \longrightarrow 2\pi - x$.

The proof can now be completed in the same way as in the infinite stair by considering first the image of a triangle with vertical sides at π and 3π and extending the mapping by Schwartz's reflection principle.

In a similar fashion we obtain a formula for the hairy plane which can be considered a polygon with infinitely many sides and interior angles alternatively equal to 2π and 0.

This means taking

$$f(z) = \int_0^z \frac{sin(x)}{cos(x)}\,dx$$

as in Figure 5

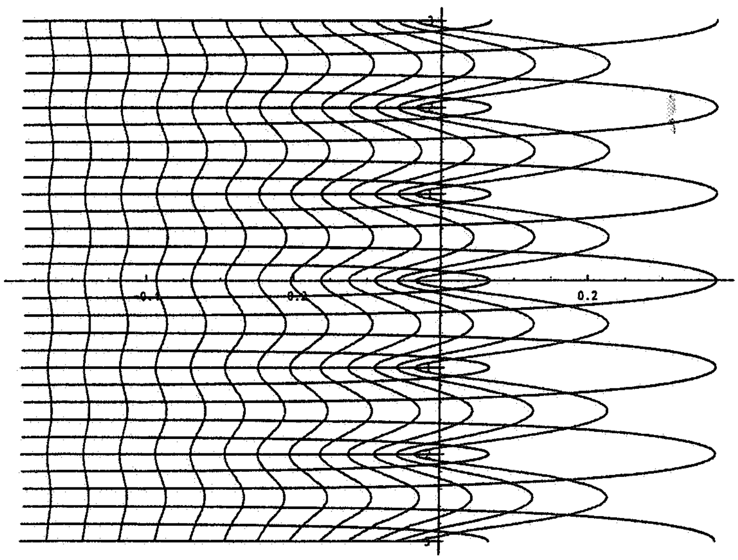


FIGURE 5. The hairy plane. $f(z) = \int_0^z tan\,\pi\,x dx$

(3) **Half a stair.**

In order to consider only one half of the stair in 1 we have to take analytic functions having half as many zeros and that implies looking at the Γ function. Indeed the formula

$$f(z) = \int_0^z \sqrt{\frac{\Gamma(1/2 - x)}{\Gamma(-x)}}\,dx$$

defines such a conformal mapping as seen in Figure 6.

To prove, however, that such a mapping is conformal we cannot use the arguments in numbers 1 ar 2 since there is no symmetry with respect to a line. We then recall the formula

$$\frac{1}{\Gamma(x)} = e^{\gamma x} x \prod_{n=1}^{+\infty} \left(1+\frac{x}{n}\right) e^{-x/n}$$

so that

$$\sqrt{\frac{\Gamma(1/2-x)}{\Gamma(-x)}} = e^{-\gamma/4} \sqrt{\frac{x}{x-1/2}} \lim_{N\to+\infty} \prod_{n=1}^{N} \left(\frac{1-x/n}{1+\dfrac{1/2-x}{n}}\right)^{1/2} e^{n/4}$$

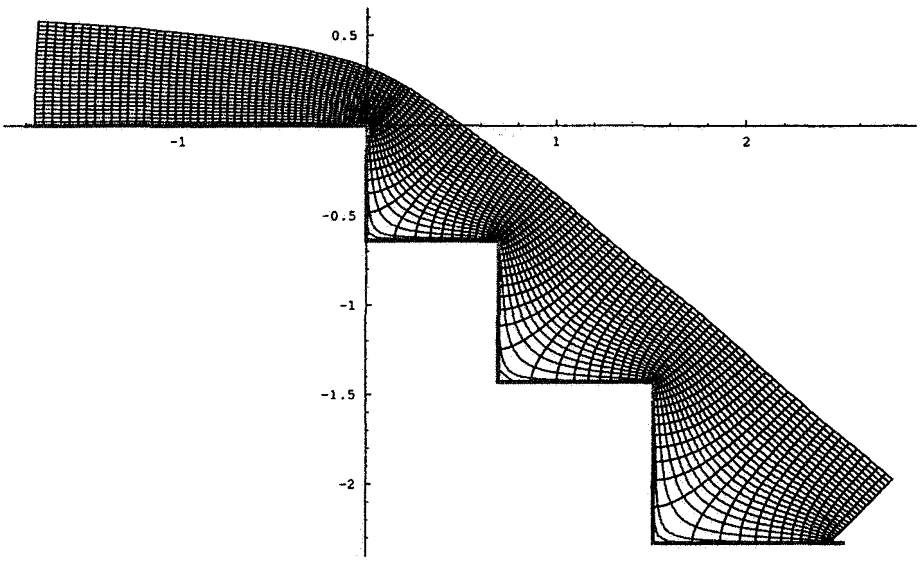


FIGURE 6. Half a stair or The stairway to the abyss. $f(z) = \int_0^z \sqrt{\frac{\Gamma(1/2-x)}{\Gamma(-x)}}\,dx$

The classical Schwarz-Christoffel formula can be applied to each finite product in the right hand side giving a stair with a finite number of steps. These conformal mappings converge uniformly to the required mapping since for each positive integer the integrals

$$\int_m^{m+1/2} \sqrt{\frac{\Gamma(1/2-x)}{\Gamma(-x)}}\,dx$$

are finite.

To obtain a half-stair with a horizontal step to the left one would naturally take

$$\int_0^z \sqrt{\frac{\Gamma(-x)}{\Gamma(1/2-x)}}\ dx$$

Finally, to obtain half a hairy plane we take

$$f(z) = \int_{-1}^z \frac{\Gamma(1/2-x)}{\Gamma(-x)}\ dx$$

as in Figure 7, or half a hairy half plane via

$$\sqrt{\Gamma(1/2-x)}\ /\ \Gamma(-x/2)$$

and so on.

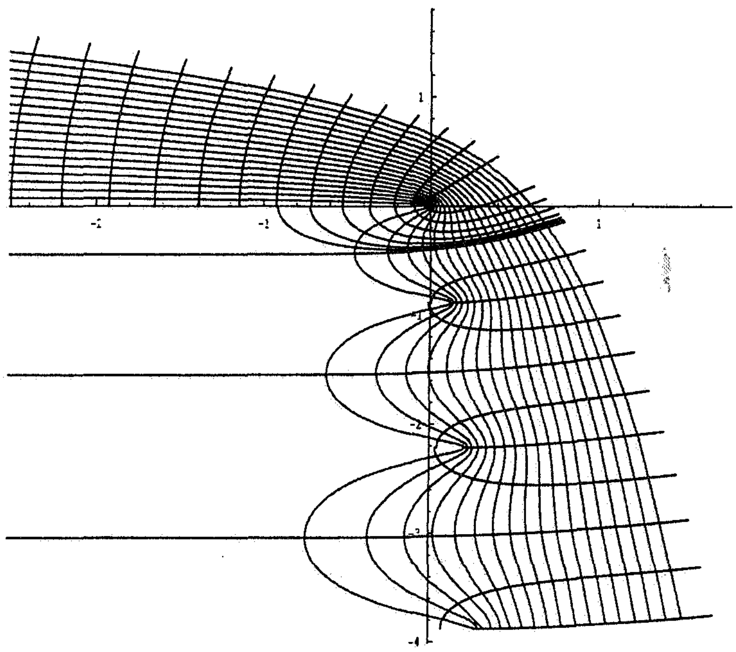


FIGURE 7. Half a hairy half plane. $f(z) = \int_{-1}^z \frac{\Gamma(1/2-x)}{\Gamma(-x)}\,dx$

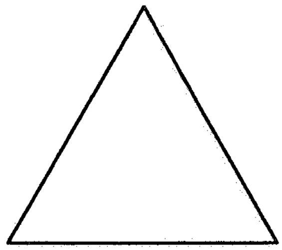
b) $F(z) = \int_0^z \frac{\Gamma(1/2-x)}{\Gamma(-x)}\,dx$

Fractals

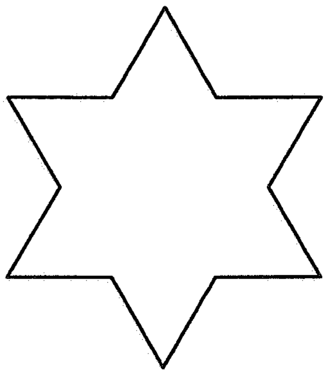
(1) Koch's snowflake

Let us recall that Koch's snowflake is obtained as the limit of a series of polygons defined as follows

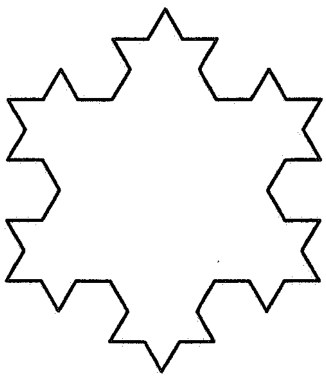
Step 0



Step 1



Step 2



and so on.

At each step we consider the Schwarz-Christoffel mapping of the unit disc into the interior of the polygon; we use the notation of the introduction throughout.

Step 0 : $\alpha_k = 1/3$, $\mu_k = 2/3$, $a_k = exp(2\pi\, i\, k/3)(k = 1, 2, 3)$

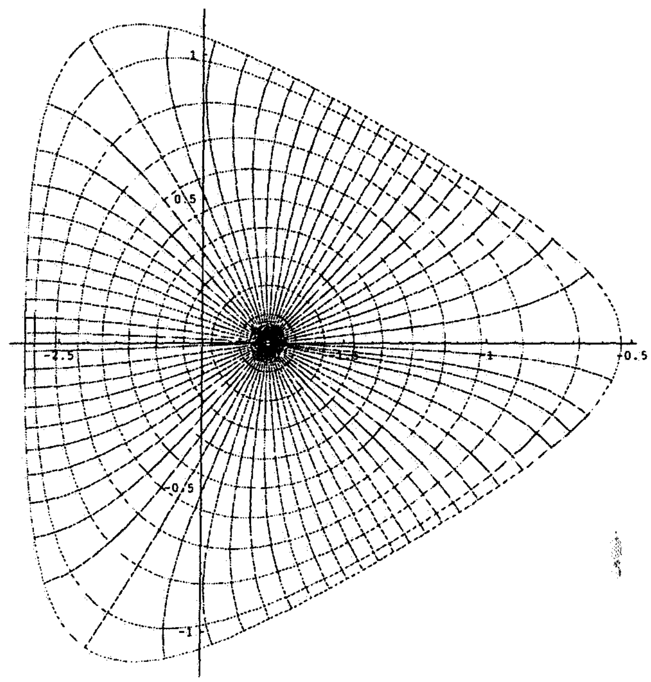


FIGURE 8. Interior of Koch's snowflake. Step 0.

$$
\begin{aligned}
F(z) &= \int_1^z \frac{dx}{(x-1)^{2/3}(x - exp(2\pi\, i/3))^{2/3}(x - exp(4\pi\, i/3))^{2/3}} \\
&= \int_1^z \frac{dx}{(x^3-1)^{2/3}}
\end{aligned}
$$

This conformal mapping is portrayed in Figure 8.

Step 1 : $a_k = exp(2\pi\, i\, k/12)$ $(0 \leq k \leq 11)$

$$
\begin{cases}
\alpha_k = 1/3\,, \quad \mu_k = 2/3 \quad k = 0, 2, 4, 68, 10 \\
\\
\alpha_k = 4/3\,, \quad \mu_k = -1/3 \quad k = 1, 3, 57, 911
\end{cases}
$$

The formula for the conformal mapping is

$$
F(z) = \int_1^z \frac{(x^6+1)^{1/3}}{(x^6-1)^{2/3}}\, dx
$$

and the image of the unit disc appears in Figure 9.

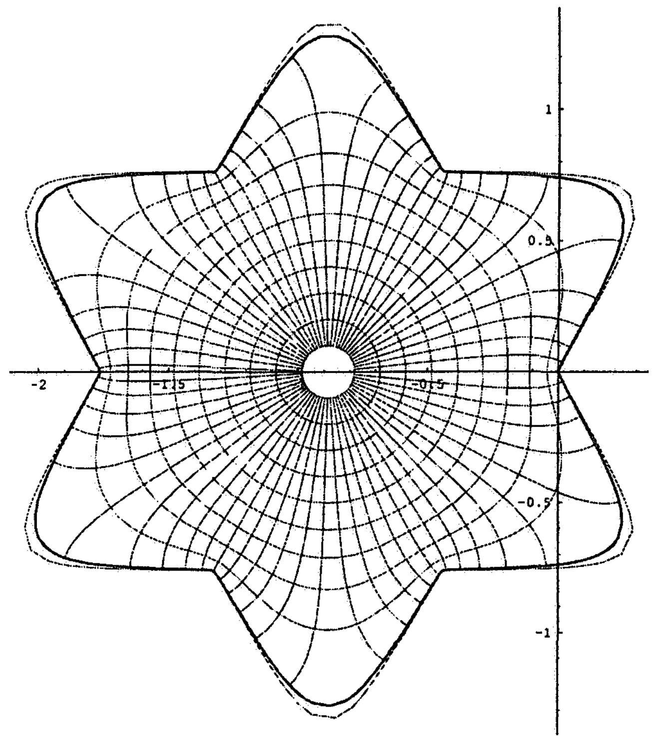


FIGURE 9. Interior of Koch's snowflake. Step 1.

Step 3 : $a_k = exp(2\pi i\, k/48) \quad (0 \le k < 47)$

$$\begin{cases} \alpha_k = 1/3, \quad \mu_k = 2/3, & k = 0,2,6,8,10,14,16,18,22, \\ & \quad 24,26,30,32,34,38,40,42,46 \\ \alpha_k = 3/4, \quad \mu_k = -1/3 & \text{all other } k \end{cases}$$

Although the combinatorics is not automatic, the formula for the conformal mapping is

$$F(z) = \int_1^z \frac{[(x^6+1)(x^{24}+1)]^{1/3}}{[(x^3-1)(x^3+1)(x^{12}+1)]^{2/3}}\, dx$$

with the image of the unit disc in Figure 10.
We may generalize

Step n : $\quad a_k = exp(2\pi i\, k/(3 \times 4^n)), 0 \le k < 3 \times 4^n$

with an integrand of the form

$$\frac{[(x^6+1)(x^{24}+1)\cdots(x^{6\cdot 4^{n-1}}+1)]^{1/3}}{[(x^3-1)(x^3+1)(x^{12}+1)\cdots(x^{3\cdot 4^{n-1}}+1)]^{2/3}}$$

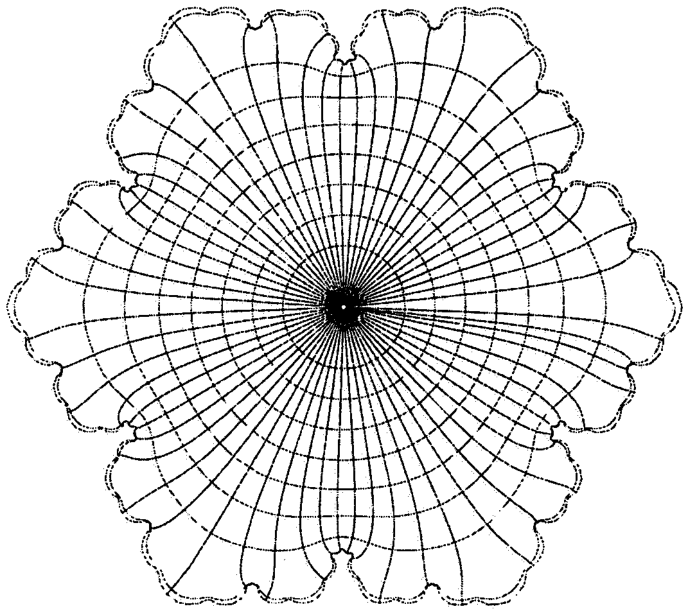


FIGURE 10. Interior of Koch's snowflake. Step 2.

Theorem:
The formula for the conformal map from the interior of the unit disc to the interior of a Koch's snowflake is

$$F(z) = \int_0^z \prod_{n=0}^{+\infty} (1 + x^{6 \cdot 4^n})\, dx\ , \ |z| < 1\,.$$

Proof.
The wording "a Koch's snowflake"

means that it is the limit of polygons with the same angles as the polygons in Koch's pattern, but with unequal sides.
First, we rewrite the integrand for step n as

$$(1+x^6)(1+x^{24})\cdots(1+x^{6\cdot 4^{n-1}})\ /\ (1-x^{3\cdot 4^n})^{2/3}$$

(module a cube root of 1) so that for $|x| < 1$ the integrand converges to the one indicated in the statement of the theorem.
The fact that the lengths are not equal to the lengths in Koch's pattern can already be seen at Step 2:

Define

$$a = \int_0^1 (1 + x^{24})^{1/3} \,/\, (1 - x^{24})^{2/3}\, dx = 1.13856$$

$$b = \int_0^1 x^6\, (1 + x^{24})^{1/3} \,/\, (1 - x^{24})^{2/3}\, dx = 0.270237$$

so that

$$F(1) = a + b$$

is one vertex of the polygon. But then

$$F(e^{\pi i/6}) = e^{\pi i/6}\, (a - b)$$

The image of a circular sector from 0 to 1 to $exp(\pi\, i/6)$ is therefore a polygon of the following form

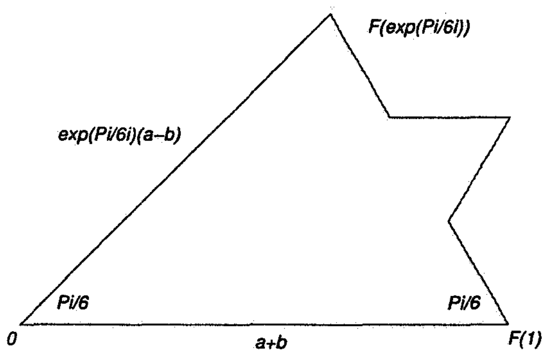


FIGURE 11.

If the lengths were equal to the regular snowflake then the vertex $F(exp(\pi\, i/6))$ would be equal to the vertex of the isosceles triangle with basis $a + b$ and basal angle $\pi/6$.

The side would equal $(a + b) \,/\, \sqrt{3}$. But

$$(a + b) \,/\, \sqrt{3} \quad < \quad (a - b)$$

showing that necessary the second length in the small triangle is shorter than the first one, leading to a location of $F(exp(\pi\, i/6))$ slightly "higher" than it should be.

Observe also that the location of this last vertex implies (the angles being as required) that the map is one-to-one on the boundary and therefore conformal throughout.

In general the mapping in step n differs from a regular snowflake, with all sides equal, but it does not differ too much.

If we analyze a sufficiently general case as in step 3 say, we have the mapping

$$f(z) = \int_0^z \quad (1+x^6)(1+x^{24}) \quad \frac{(1+x^{96})^{1/3}}{(1-x^{96})^{2/3}}\, dx\,.$$

It is easy to obtain the points

$$\begin{aligned}&P^* = f(1) = 1.35 \quad , \quad Q^* = f(e^{\pi\, i/24})\,,\\ &R^* = f(e^{\pi\, y/8}) \quad , \quad S^* = f(e^{\pi\, i/6})\,.\end{aligned}$$

If we compare them to a regular snowflake with vertices P, Q, R, S we obtain points at a distance smaller than 0.1

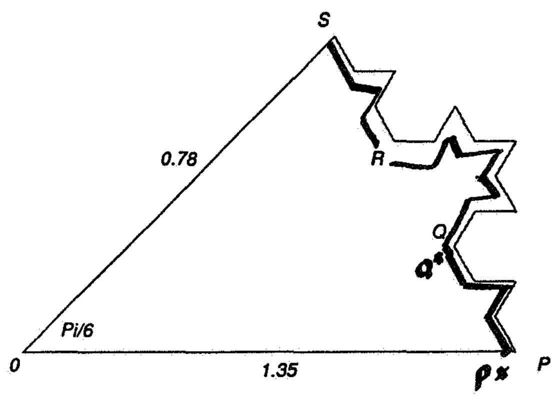


FIGURE 12.

what is to be noticed is that for the image not be one-to-one on the boundary the point Q^* should differ largely from Q, to be below PO for example; the position of P^*, Q^*, R^*, S^* forces then the actual polygon to be free of self-intersections, the angles being $\pi/3$ or $2\,\pi/3$. In dotted line we draw such a possible polygon in Figure 12.

The sequence of conformal converges then to a non-constant (since $F'(0) = 1$) conformal map.

We check that the image is not all of $\mathbb{C}$.

Claim:

$$F(1) = \int_0^1 \prod_{n=0}^{+\infty} (1 + x^{6 \cdot 4^n}) \quad dx \leq 3/4$$

Indeed, with $y = x^6$ we have $0 \leq y \leq 1$ and

$$\begin{aligned}(1+y)(1+y^4)(1+y^{16}) \cdots &= \quad 1 + y + (y^4 + y^5) + \\ &\quad (y^{16} + y^{17} + y^{20} + y^{21}) + \cdots \\ &\leq 1 + y + 2y^4 + 4y^{16} + \cdots \\ &= 1 + \sum_{n=0}^{\infty} 2^n \, y^{4^n}\end{aligned}$$

Thus the integral is bounded from above by

$$\int_0^1 \quad 1 + \sum_{n=0}^{+\infty} \quad 2^n \, x^{4^n \cdot 6} dx \quad = \quad 1 + \sum_{n=0}^{+\infty} \quad \frac{2^n}{6 \cdot 4^n}$$

$$\leq 1 + \frac{1}{6} \quad \sum_{n=0}^{+\infty} \quad \frac{1}{2^n} = \frac{4}{3}$$

We also check that the image is not a disc.
Set

$$a = \int_0^1 \prod_{n=1}^{+\infty} (1 + x^{6 \cdot 4^n}) \, dx, \quad b = \int_0^1 \quad x^6 \quad \prod_{n=1}^{+\infty} (1 + x^{6 \cdot 4^n}) \, dx$$

Then $F(1) = a + b$ and $0 < b < a$.

But

$$F(e^{\pi i/6}) = e^{\pi i/6}(a - b)$$

has modulus smaller than $F(1)$.

Finally we have to explain a subtle point that appears when we draw a figure such as Figure 10.

One cannot write for example

$$(1 + x^6)^{1/3}(1 + x^{24})^{1/3} = [(1 + x^6)(1 + x^{24})]^{1/3}$$

since the branch we implicitly use by analytic continuation may not coincide with the fixed branches of the mathematical software.

With

$$u(k,x) = log(exp(\pi\, i\, k/24) - x)$$
$$v(k,x) = log(x - exp(\pi\, i\, k/24)$$

then the numerator of the integrand is

$$\begin{aligned}\mathrm{Num}(x) \;=\;& exp\,(1/3(u(1,x)+u(3,x)+u(3,x)+u(4,x)+u(5,x)\\ &+u(7,x)+u(9,x)+u(11,x)+v(12,x)+v(13,x)\\ &+v(15,x)+v(17,x)+v(19,x)+v(20,x)+\\ &v(21,x)+v(23,x)+v(25,x)+v(27,x)+\\ &+v(28,x)+v(29,x)+v(31,x)+v(33,x)+v(35,x)+\\ &+v(36,x)+u(37,x)+u(39,x)+u(41,x)+u(43,x)\\ &+u(44,x)+u(45,x)+u(47,x)))\end{aligned}$$

and for the denominator

$$\begin{aligned}\mathrm{Den}(x) \;=\;& exp(2/3(u(0,x)+u(2,x)+u(6,x)+\\ &+u(8,x)+u(10,x)+v(14x)+v(16,x)+\\ &+v(18,x)+v(22,x)+v(24,x)+v(26,x)+\\ &+v(30,x)+v(32,x)+v(34,x)+u(38,x)+\\ &+u(40,x)+u(42,x)+u(46,x)))\,.\end{aligned}$$

At any rate, the formula of the theorem with terms of up to $6 \cdot 4^6$ is sufficiently precise (and simpler to use) and gives Figure 10.

(2) **The exterior of the snowflake.**

At Step 0 we use the formula of the introduction with $a_k = exp(2\pi\, i/3 \times k), \mu_k = 2/3, \quad 1 \le k \le 3$.

In this case

$\lambda = 2/3 \sum_{k=1}^{3} d_k = 0$ so that the formula is

$$F(z) = \int_1^z (x^3-1)^{2/3} \frac{dx}{x}^2$$

whose image is in Figure 14.

In the same way, the formulas for steps 1 and 2 are

$$\int_1^z \frac{[(x^3-1)(x^3+1)]^{2/3}}{[(x^6+1)]^{1/3}} \;\; \frac{dx}{x^2} \qquad \text{and}$$

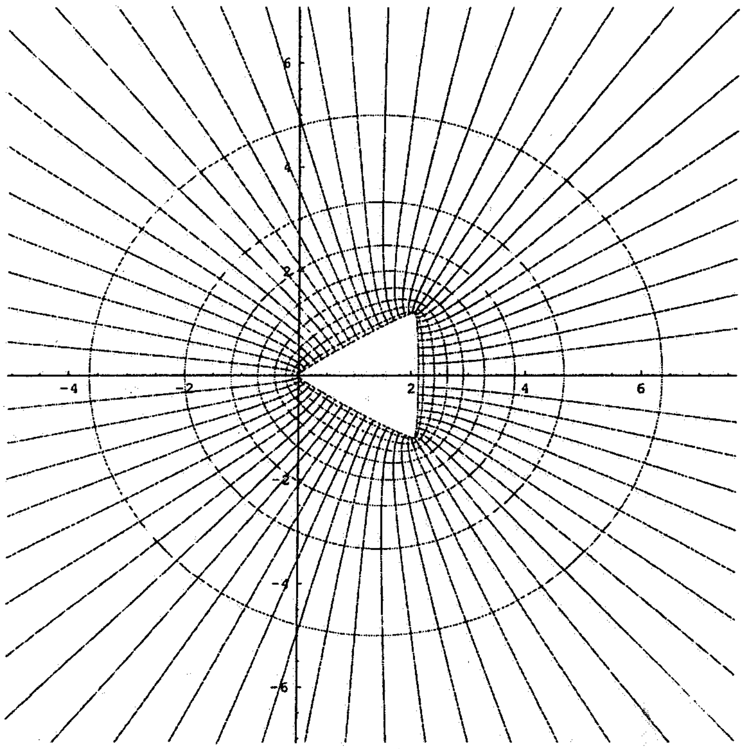


FIGURE 13. Exterior of Koch's snowflake. Step 0.

$$\int_1^z \frac{[(x^3-1)(x^3+1)(x^{12}+1)]^{2/3}}{[(x^6+1)(x^{24}+1)]^{1/3}} \quad \frac{dx}{x^2}$$

corresponding to the Figures 15 and 16.

The mapping from the unit disc to the exterior of Koch's snowflake is

$$F(z) = \int_1^z \left(\prod_{n=0}^{+\infty} (1 + x^{6 \cdot 4^n}) \right)^{-1} \quad \frac{dx}{x^2}$$

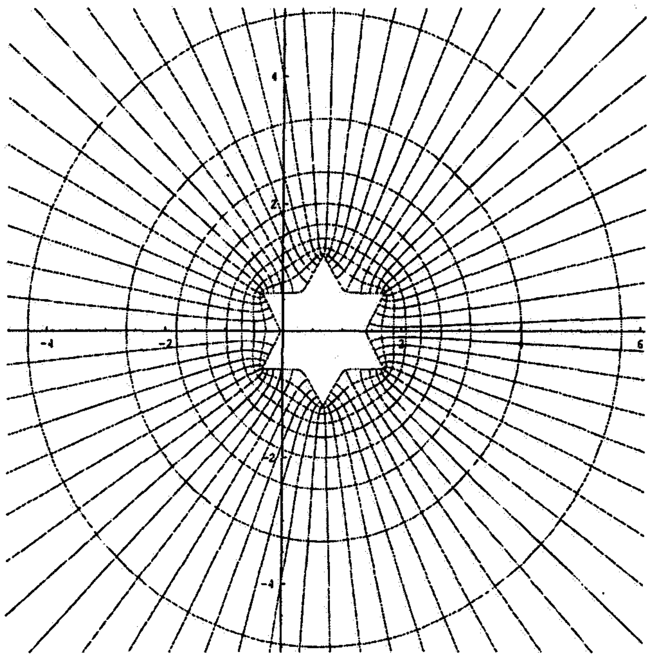


FIGURE 14. Exterior of Koch's snowflake. Step 1.

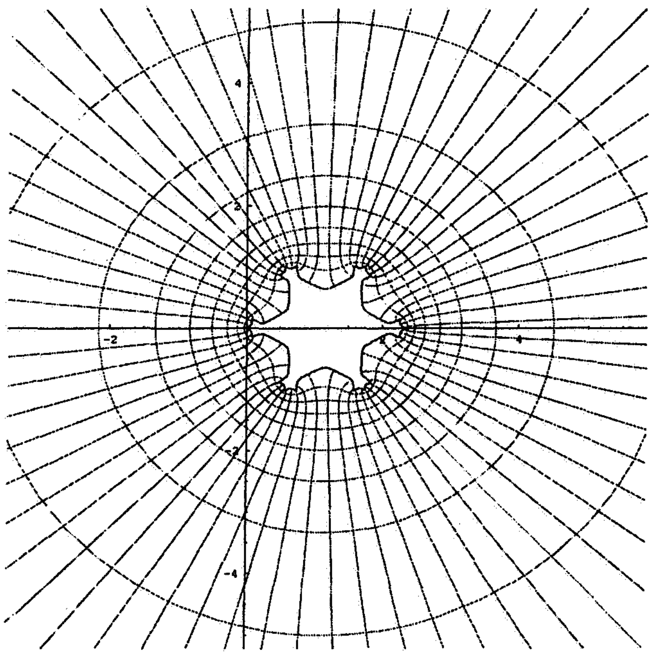
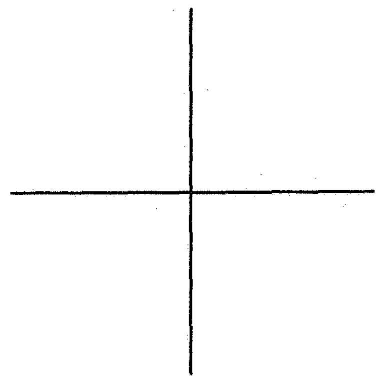


FIGURE 15. Exterior of Koch's snowflake. Step 2.

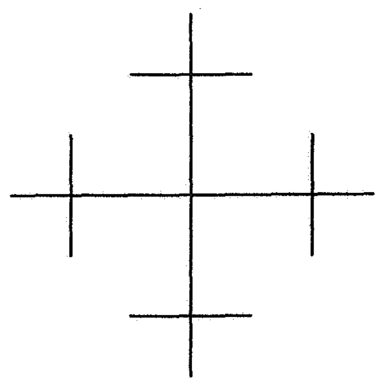
(3) **A tree.**

We define a sequence of "trees" as follows

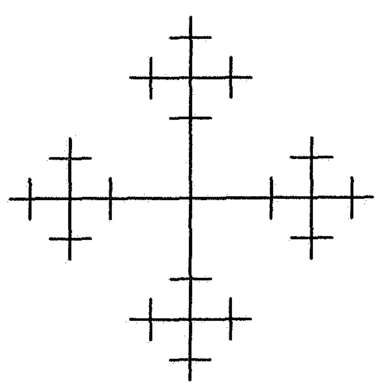
Step 0



Step 1



Step 2



and so on.

We want to define accordingly a sequence of Schwarz-Christoffel conformal mappings from the unit disc to the complement of these trees.

The internal angles al step 0 give $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = 0$ with $\mu_1 = \mu_3 = \mu_5 = \mu_7 = 1$ at the points and $\alpha_2 = \alpha_4 = \alpha_6 = \alpha_8 = 3/2, \quad \mu_2 = \mu_4 = \mu_6 = \mu_8 = -1/2$ at the corners.

Therefore

$$F(z) = \int_1^z \frac{1+x^4}{(1-x^4)^{1/2}} \frac{dx}{x^2}$$

is the required formula at step 0.

At step 1 we obtain.

$$F(z) = \int_1^z \frac{1+x^4}{(1-x^4)^{1/2}} \frac{1+x^8}{(1+x^{16})^{1/2}} \frac{dx}{x^2}$$

with branch at the 32 roots of unity.

Similarly at step 2

$$F(z) = \int_1^z \frac{1+x^4}{(1-x^4)^{1/2}} \frac{(1+x^8)(1+x^{32})}{[(1+x^{16})(1+x^{64})]^{1/2}} \frac{dx}{x^2}$$

The image is portrayed in Figure 17.

At the limit we obtain the formula

$$F(z) = \int_1^z \frac{1+x^4}{(1-x^4)^{1/2}} \frac{\prod_{n=1}^{+\infty}(1+x^{2\cdot 4^n})}{[\prod_{n=1}^{+\infty}(1+x^{4^{n+1}})]^{1/2}} \frac{dx}{x^2}$$

for the conformal mapping from the unit disc to the exterior of this fractal.

There are of course many more trees to be considered but the combinatorics arising from angles and points is not straightforward so that it may be hard to obtain a general expression for the conformal map. We leave these as questions for further study.

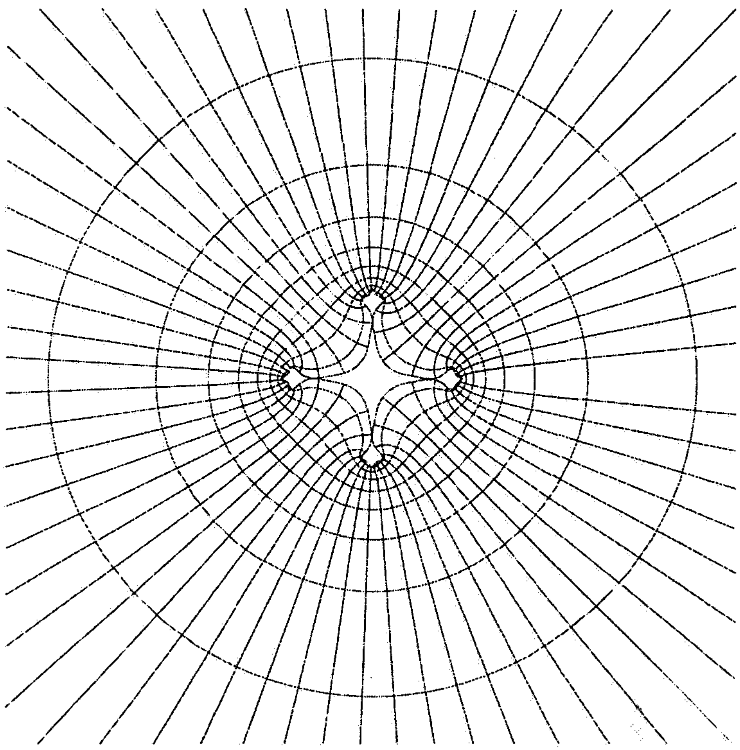


FIGURE 16. Exterior of a tree. Step 2.

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