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ABSTRACT. We prove that any 3-manifold can be obtained from a 3-ball that is conveniently triangulated and 4-colored. A ball like this is called a butterfly. As a corollary we obtain a theorem of Izmestiev and Joswig.

Key words and phrases. Knots, Fundamental group, 3-manifolds, Triangulations, Branched coverings.

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1. INTRODUCTION

Recently, I. Izmestiev and M. Joswig ([6] and [5]) have shown that for any triangulation Δ in a manifold N, there is an associated group $\Pi(\Delta)$, called the group of projectivities of Δ. This group has some similarities with the fundamental group, even though it is not a topological invariant. In fact, the action of it on the set of vertices of a simplex of Δ, permits the construction of branched coverings over N. In this way, they show that any closed orientable 3-manifold M arises as a branched covering over S^3 from some triangulation of S^3. Their proof uses the theorem that asserts that any closed orientable 3-manifold M is a simple 3-branched covering over S^3 with a knot K as branched set (see [3] and [8]). They start from a tubular neighbourhood R of the knot K and give a triangulation for it. Now, using handlebody decomposition, they attach triangulated k-handles to finally find the triangulation Δ of S^3 from which M arises as a branched covering of S^3. Unfortunately, their proof is not constructive. Nevertheless, this result is important because it shows how to consider branched coverings in a combinatorial way.

In this paper we approach the problem from another point of view (see [11]). It is known that any 3-manifold can be obtained as a ball with identified faces. Here we study very special kinds of balls with identified faces that we call butterflies. We show that for any knot diagram there is one butterfly that gives S^3, once the faces are identified. Then we show that for any knot that admits a representation of its fundamental group in the symmetric group of 3 elements, and that sends the meridians to transpositions, it is possible to triangulate algorithmitically the given ball, and to identify its faces so as to obtain the desired triangulation of S^3.

Now, since from these triangulation we obtain a 3-manifold M, a natural triangulation arises in M and hypothetically all the machinery of combinatorial topology could be used to identify M. And using the results of A. Thompson ([12], [13] and [14]) there is hope to be able to prove or disprove the Poincaré Conjecture.

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2. Knots and Butterflies.

In this section, we study a special class of balls with faces which are identified by topological reflections. These balls are called butterflies since the identifications of each pair of faces recall how a butterfly closes its wings. Here are formal definitions.

Let F be a connected, closed, orientable surface. A *polygon* (or n-gon, $n \geq 1$) in F is a tame embedding of the 2-disk in F together with a set of $n \geq 1$ points in its boundary which are called the *vertices of the polygon.* The closures of the connected components of the complement of the vertices in the boundary of a polygon are called the *edges of the polygon.* (An edge is an arc if $n \geq 2$ or a circle if $n = 1$.) A *polygonization* of F is a decomposition of F in a union of a finite number of polygons such that (i) the interiors of the distinct polygons of the decomposition are disjoint; and (ii) if two arbitrary polygons intersect, their intersection is at the same time a union of vertices and edges, and a connected 0-, or 1-dimensional manifold. (Therefore they can intersect in just one vertex, or in an arc formed by various edges, or in a circle; and, in this case, F must be S^2.) The union R of the boundaries of the polygons of the decomposition is then a connected graph embedded in F, since it is a union of vertices and edges. We say that *the graph R poligonizes F.*

Example 1. *Figures 1 and 6 show different poligonalizations of S^2.*

Let A and B be two polygons of a polygonization of S intersecting in exactly one edge α of R and assume that A and B have the same number of edges. (This number might be one for the trivial polygonization of S.) Select a topological reflection $\alpha : A \to B$ which is orientation reversing in S, fixes each point of α, and sends vertices (resp. edges) of $R \cap A$ into vertices (resp. edges) of $R \cap B$. The reflection along α will be denoted by α also and we say that $\alpha : A \to B$ is an α-reflection.

Definition 1. *Given $n \in \mathbb{N}$, $n \geq 1$, an n-butterfly (B, R, T) is a 3-ball B with a polygonization of its boundary S by a graph R into $2n$ polygons, together with a subset T of n mutually disjoint edges of R, such that the polygons are identified by α-reflections in pairs, $\alpha \in T$. (To be identified, two faces must share exactly an edge $\alpha \in T$. The identification of a pair of faces is then achieved by an α-reflection along this common edge.) The $2n$ polygons of an n-butterfly are called wings, the union of the edges along which we made the reflections is called the trunk T and n is called the butterfly number.*

The result of the identification of pairs of wings of an n-butterfly (B, R, T) is a 3-pseudomanifold $M(B, R, T)$ (See, for instance [10].). Denote by $p : B \to M(B, R, T)$ the natural projection.

Definition 2. *In the identification $p : B \to M(B, R, T) = S^3$ the image $p(T)$ of the trunk T is a (linked) graph K embedded in S^3, and we say that the graph (knot, or link) K in S^3 can be represented by an n-butterfly, or that K admits an n-butterfly representation, or that (B, R, T) is an n-butterfly representation of K.*

Let (B, R, T) be a butterfly and $p : B \to M(B, R, T) = S^3$t he natural projection. We now make a classification of the set of vertices of R. A vertex V of R will be called an A-vertex iff $V \in T$. (It will be generically represented by A.) The vertex V of R will be called an E -vertex iff it is not an A-vertex but $V \in p^{-1}p(A)$. (It will be generically represented by E.) Finally the vertex V of R will be called a B-vertex iff $V \notin p^{-1}p(A)$ for any A-vertex A. (It will be generically represented by B.)

The set $p^{-1}p(V)$ will be denoted by $\{V\}$. The set of $\{V\}$ classifies the set of vertices of R.

In the sequel we will consider definitions and concepts about knots, links and their projections as explained in [1].

It is obvious that the trivial knot is the only one admitting a 1-butterfly. representation (just identify the northern and southern hemispheres of S by reflection in the equator $R = T$).

Let us recall that for each rational number p/q $(p > q > 0)$ there is a knot denoted by the same number p/q.

The following theorem is a translation of [9, page 164] to our language of butterflies.

Theorem 1. *Every 2-bridge knot or link p/q can be represented as a 2-butterfly. Except, for the trivial knot or the link $2/1$ (Hopf link) the butterfly will have E-vertices.*

Proof. For shortness we give the proof only for the rational knot 5/3. Without difficulty, it could be generalized to every other 2-bridge knot or link. We start from the following 2-butterfly and we show that performing the reflections illustrated by arrows in the figure, we obtain the knot $\frac{5}{3}$.

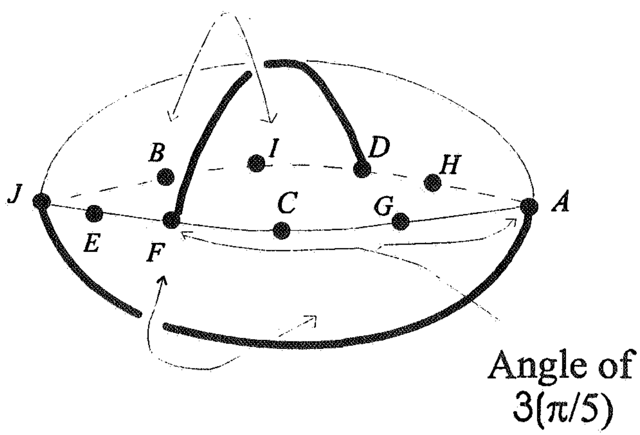


Figure 1

This butterfly has four wings; $FDHAGC$; $FEJBID$; $JEFCGA$ and $AHDIBJ$, which are identified two by two by the reflections made along the edges FD and AJ, that are indicated by arrows. The trunk is $\{FD, AJ\}$. The vertices C, G, H, I, B and E are E-vertices.

Now, we make a sequence of deformations in order to be able to visualize the knot $\frac{5}{3}$. First of all, we stretch out the points placed on the butterfly "equator", obtaining a cylinder. Next we rotate the "upper lid" of the cylinder an angle $3(\frac{\pi}{5})$.

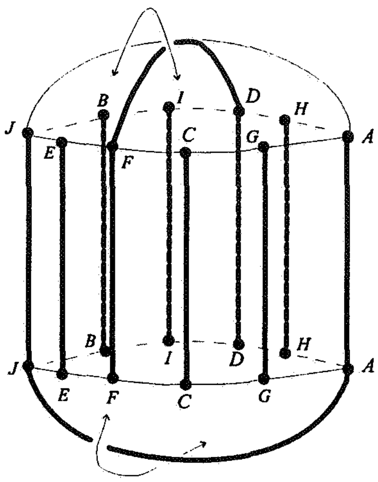


Figure 2a

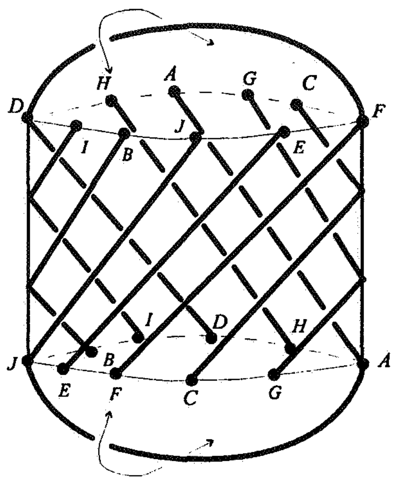


Figure 2b

Finally, making the identifications indicated by the arrows on the "upper and lower lids" of the cylinder, the knot $\frac{5}{3}$ becomes visible.

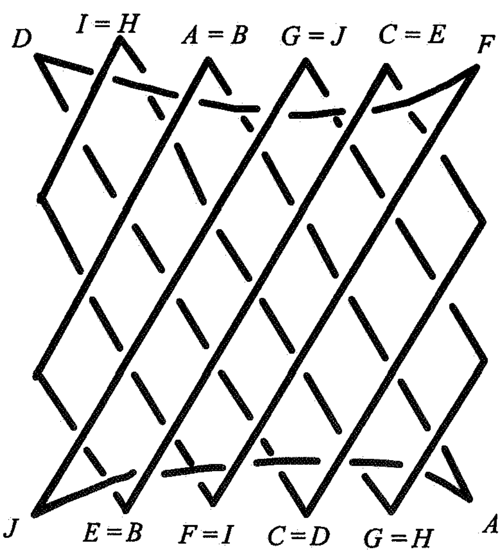
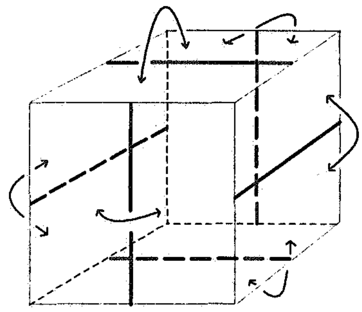


Figure 3

□

Example 2. *Similarly Thurston showed in* [15] *that the Borromean rings admits a* 6 *-butterfly representation. In this butterfly there are 12 A-vertices and 8 B-vertices. See Figures 4a and 4b.*



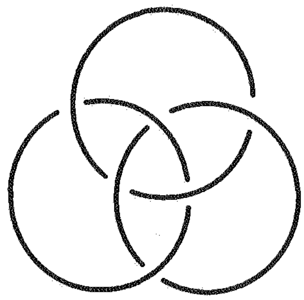


Figure 4a *Figure 4b*
A butterfly for the Borromean rings.

3. Representing knots as butterflies.

Actually, every knot or link can be represented by an n-butterfly, for some $n \in \mathbb{N}$. This representation will depend on the diagram of the knot or link, so for each knot or link we have an infinite number of butterfly representations.

Theorem 2. *Every knot or link admits an n-butterfly representation, for some* $n \in N$.

Proof. There is a more detailed proof in [4]. Let K be a knot (or link) which is embedded in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, we assume that it is as flat as possible as in the following picture.

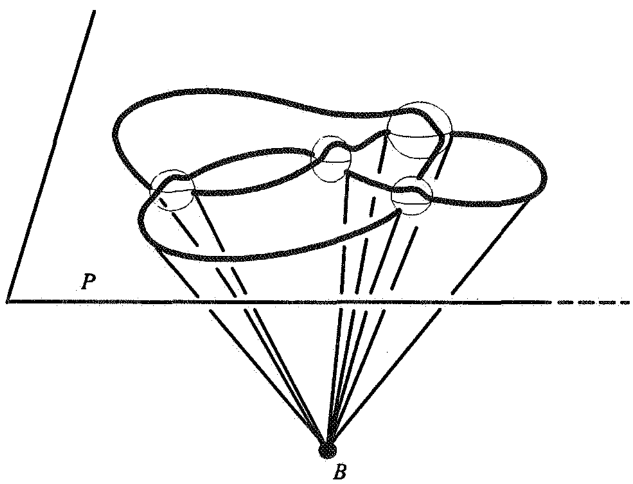


Figure 5

To get a butterfly that represents the knot we cut along a cone of the knot. The next picture illustrates the boundary of the 4-butterfly representation of the eight knot. We envision that the interior of the butterfly is placed over the paper. We can see 20 edges. Three of them go to the point at infinity. The trunk has 4 edges that form the knot before cutting the cone off and we have 8 wings identified in pairs.

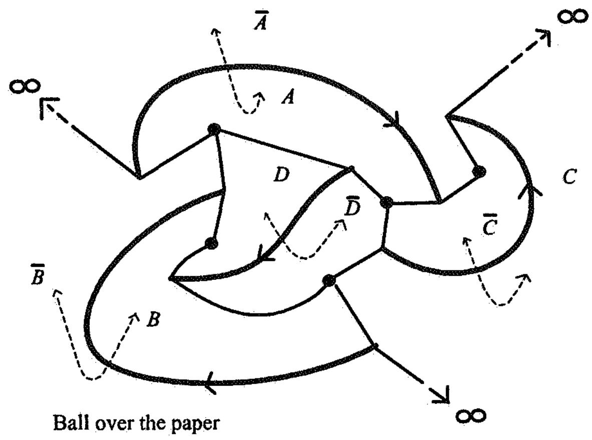


Figure 6

□

A regular diagram of a knot (or of a link), like the one in Figure 5 or 7a, can be thought of as a disjoint union of arcs in a plane. These arcs will be called the *arcs* of the diagram.

We remark that the last theorem is constructive. Now, we make explicit the algorithm for constructing a butterfly from a given regular diagram.

Algorithm for constructing a butterfly associated to a diagram of a knot:

Let D_K be a regular diagram of a knot (or link) K , which we assume oriented in order to fix notation. Without lost of generality we can assume that K is not the trivial knot (for this case we have a 1-butterfly representation).We also assume S^3o riented. The positive orientation of S^3in our figures will be given by a right handed screw.

Step 1. The diagram D_K of K is a finite collection T of disjoint (oriented) arcs in a plane P (The plane of the paper; see Figure 7a. The projection of D_K onto P has no kinks and is connected.).

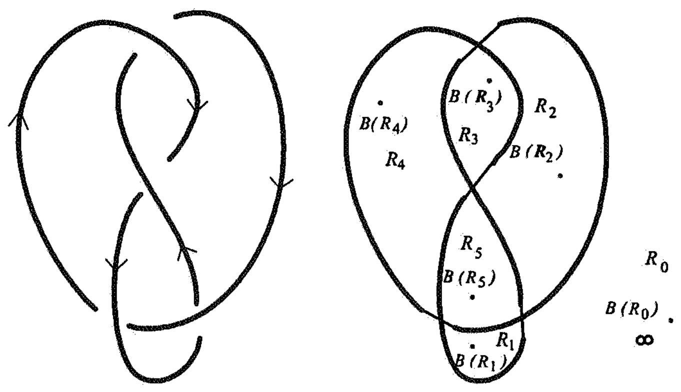


Figure 7a Figure 7b

Step 2. We consider the regions R_i determined in the plane P by the projection of K (See Figure 7b.). In the interior of each bounded region we choose a point and label it by $B(R_i)$. For the unbounded region R_0, we label $B(R_0)$ the point at the infinity. These points are B-vertices.

Step 3. Each end (or A-vertex) of an arc α of the collection T is joined, using an arc in P, with each of the B-vertices that belong to the adjacent regions to the arc α. So the paper becomes polygonalized by the graph R which is the union of the trunk T and the added arcs. (See Figure 6.)

Step 4. Over the plane of the paper P, we assume that there is a 3-ball D_K^*, with the induced orientation, whose boundary is the polygonalized plane P, oriented as the boundary of the oriented 3-ball D_K^*.

Step 5. The adjacent faces to each arc α of S are identified by an α-reflection that is indicated by double arrows. We denote by $\bar{A}$ the wing (or face) identified with the wing A. The face A will be placed at the right side of the oriented arc α. (See Figure 6.)

Then (D_K^*, R, T) is the wanted butterfly. It will be called the *butterfly associated to the diagram D_K of the knot K*. For shortness, we denote it simply by D_K^*.

For each arc α of the trunk T, there are two adjacent wings in D_K^* that are identified by a reflection along α and they have the following shape:

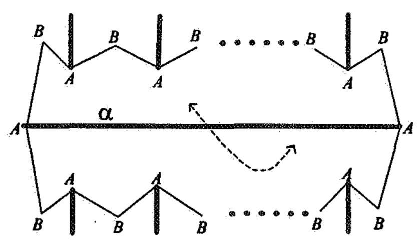


Figure 8

On each wing we distinguish two types of vertices:

A vertices are the ends of the differents arcs of T.

B vertices are those that come from the vertex of the cone once we cut it off. They correspond to the points $B(R_i)$ given in the algorithm.

Example 3. *Applying this algorithm to the regular diagram of the Borromean rings shown in Figure 4b, we recover the butterfly representation discovered by Thurston (Figure 4a).*

4. Painting butterflies.

As in the precedent section, we will assume that the butterfly is oriented; its boundary has the induced orientation, and the trunk of the butterfly is oriented. If α is an edge of the trunk, its adjacent faces are A at its right side, and $\bar{A}$ at its left side.Recall that the reflection along α is denoted by α.

Definition 3. *If each (oriented) edge of the trunk of a butterfly is endowed with a permutation we say that the butterfly is painted.*

If (B, R, T) is a butterfly representation of a knot K, each (oriented) edge α of the trunk T defines a meridian generator of the knot group of K as follows. Take an interior (base) point O inside B and run an oriented arc from O to an interior point in A and another oriented arc from the corresponding point in $\bar{A}$ back to O. The (oriented) union of these two arcs represents a meridian generator of the knot group that will be denoted by α^*. If ω is a representation of the knot group into the symmetric group Σ_n of the numbers $1, 2, ..., n$ we can endow the (oriented) edge α of the butterfly with the permutation $\omega(\alpha^*)$. (If the representation ω sends meridians to elements of Σ_n of order two, then the permutations $\omega(\alpha^*)$ and $(\omega(\alpha^*))^{-1}$ coincide and the orientation of the trunk becomes irrelevant.) In this way, we have obtained a painted butterfly, denoted by (B, R, T, ω).

In the sequel we will assume that for $n < m$ there is a natural inclusion of Σ_n as a subgroup of Σ_m induced by the inclusion $\{1, 2, ..., n\} \subset \{1, 2, ..., m\}$. In this way an element of Σ_n acts in the set $\{1, 2, ..., m\}$ fixing the numbers $\{n + 1, ..., m\}$.

Let Δ be a triangulation of a compact connected surface M^2, with or without boundary. an n-*coloration of* Δ is a function $C : V \to \{1, 2, ..., n\}$, where V is the set of vertices of Δ such that the three vertices of each triangle of Δ have three different assigned colors. The numbers $1, 2, ..., n$ are called the *colors*. (an n-coloration is not necessarily surjective. In this way any n-coloration is trivially an n+1-coloration.)

We will understand that the boundary of a butterfly is *triangulated* if we have a triangulation for the butterfly boundary, such that all the triangles become identified by couples when we identify the butterfly wings using the reflections along the edges of the trunk.

If ω is a representation of the knot group of K into the symmetric group Σ_n, (B, R, T) is a butterfly representation of K and the boundary of the corresponding painted butterfly (B, R, T, ω) is triangulated by Δ and colored by $C : V \to \{1, 2, ..., m\}$, $m > n$, we say that C is *compatible with* ω iff under the α-reflection (α is the arc shared by A and $\bar{A}$) the color k matches with color $\omega(\alpha^*)(k)$, that is, for every vertex $v \in A$ we have $C(\alpha(v)) = \omega(\alpha^*)(C(v))$.

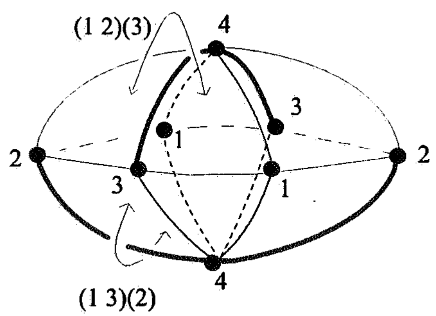


Figure 9

Example 4. ***A painted butterfly for the knot 3/1.*** *The Figure 9 illustrates a triangulation* Δ *of a butterfly* (B, R, T) *that represents the rational knot* $3/1$ *(trefoil knot). It is 4-colored by* C *as shown. It turns out that the map of the group of* $3/1$ *into* Σ_3 *sending the two generating meridians* α^* *and* β^* *(associated to the trunk* T*) to the permutations* $(12)(3)$ *and* $(13)(2)$*, respectively, is a homomorphism* ω *(representation) into* Σ_3*. It is easy to see that the coloration* C *is compatible with the representation* ω*.*

5. Main Theorem

Theorem 3. *Let K be a knot with a given regular diagram D_K. Suppose that $\omega : \pi_1(S^3 \backslash K) \to \Sigma_3$ is a representation sending meridians to transpositions. Then there is a triangulation Δ of the boundary of the butterfly D_K^* and a 4-coloration of Δ which is compatible with ω.*

Proof of the Main Theorem. Our purpose is to find a triangulation of the butterfly boundary and a 4 -coloration compatible with the representation ω. In other words, we need a triangulation of the boundary of the butterfly D_K^* such that if two triangles are identified by an α-reflection whose associated transposition is $(i\ j)$, $i, j \in \{1,2,3\}$, the α-reflection sends colors (i,j) to colors (j,i), and leaves the remaining two colors fixed. (The number 4 remains always fixed.) In particular, in an α-reflection with permutation $(i\ j)$ the vertices upon the arc α are colored with one of the remaining numbers.

First of all, we construct an initial triangulation Δ, and we assign four colors to the vertices in such a way they are compatible with ω. After that we will refine Δ in order that colors along adjacent vertices be different.

Let us describe the set of vertices of Δ. This set contains all the A and B vertices given in the Algorithm before. Moreover, it also contains a point, which is taken in the interior of each arc of the diagram. We denote generically these points by D and we say that they are *D vertices.*

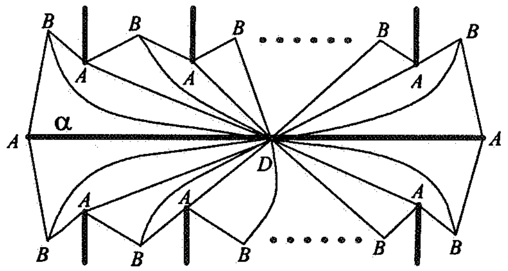


Figure 10

Now, we describe the edges of Δ. On one hand, the curves AB and AD that are contained in the boundary of the wings are edges of Δ. On the other hand, we notice that each wing has only one D vertex in its boundary. From that point we trace disjoint curves (except in D) toward every B and A-vertex (that is not an end of the arc that contains the point D). These additional curves are contained in the same wing that contains the point D and are also edges of Δ by definition.

We already have the triangulation Δ. We remark that each triangle has one vertex of each type.

Now, we color the vertices of Δ. In order to get a 4-coloration compatible with ω, first of all, we color the A and D vertices which are the only ones lying in the edges α of the trunk. If the permutation associated to the edge α is $(i\ j)$, the vertices of α are to be given a color in $\{1,2,3,4\} - \{i,j\}$.

Assign the number 4 to D vertices and the color $\{1,2,3\} - \{i,j\}$ to A vertices. The color 4 given to D vertices is compatible with ω because ω fixes 4. We observe that the colors given to the A vertices are also compatible with ω. In fact, this is trivial for the A-vertices that are ends of the arc α; they remain fixed under the

α-reflection. Hence, let us consider an A vertex that belongs to the adjacent wings to α but is not an end point of α. Its corresponding point under the α-reflection is another A vertex, and these two A vertices arise from the same crossing point of the diagram. So, their colors also match under the permutation $(i\ j)$ associated to α because ω associates to each arc a transposition in Σ_3 and the only possibilities for the A vertices that arise from the same crossing point are the following: A vertices belong to arcs with the same associated transposition (Figure 11a) or belong to arcs with different associated transpositions (Figure 11b). (Actually, the fact that the colors of the A-vertices match is a consequence of the Wirtinger relations that are verified at each crossing point of the diagram.)

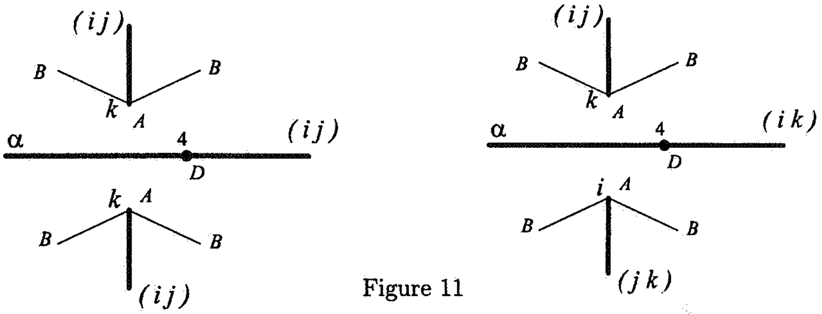


Figure 11

Now, we proceed to color B vertices. Recall that for each region R_i, determined by the projection of the knot, there is exactly one B vertex denoted by $B(R_i)$.

We note the obvious fact that starting in a region R_i, it is possible to tour all the regions going transversally across the arcs of the knot. For instance, here is a curve visiting all the regions: Our initial point is a point Q in a given region. Then we move following a parallel curve to the knot until we again find Q. Once there, we traverse the knot and we choose another point in the adjacent region to the initial one, that we label R. We continue touring the regions moving along a parallel curve to the knot until we find R and we stop there. In this way we have passed through every region because every region is bounded by the knot.

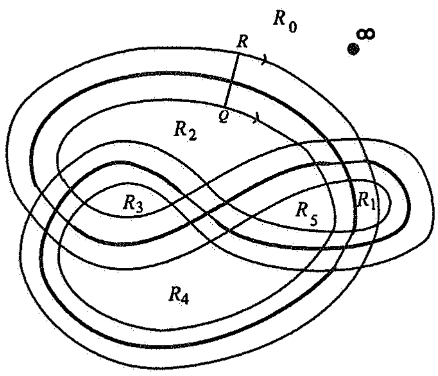


Figure 12

Now, let us give the coloring rule for the B vertices. We start by assigning any of the numbers 1, 2 or 3 to one of them. Once it is colored the others become colored in the following way: if R_{i_1} and R_{i_2} are two regions that share an arc to which is associated the transposition $(i\ j)$ and if we have assigned the color $k \in \{1,2,3\}$ to $B(R_{i_1})$, then the vertex $B(R_{i_2})$ gets the color $(i,j)k$, i.e., the image of k under the transposition $(i\ j)$.

This coloring rule and the fact that starting in any region we can travel to any other region guarantees that it is enough to choose the color of only one of the $B(R_i)$ to get all the B vertices colored. Moreover, since we have three options for coloring the first one, each of the three gives a different coloration of the $B(R_i)$ vertices. (Incidentaly, we will see later that these different colorations might give rise to different triangulations.)

We need to prove that given the color of $B(R_i)$, then the color of $B(R_j)$ is independent of the chosen path between $B(R_i)$ and $B(R_j)$. In fact, it is enough to observe what happens with the colors of B vertices of the four regions sharing a crossing point.

Let $R_{i_1}, R_{i_2}, R_{i_3}$ and R_{i_4} be the four adjacent regions to a fixed crossing point. If we give the color, for example, to $B(R_{i_1})$ then the colors of $B(R_{i_2}), B(R_{i_3})$ and $B(R_{i_4})$ are determined. In fact, the two possible cases are:

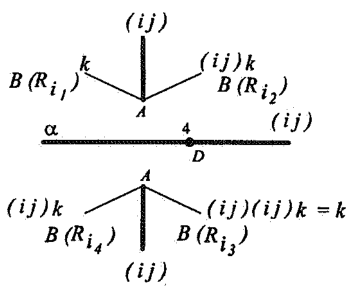


Figure 13a

1. The three arcs have the same associated transposition.

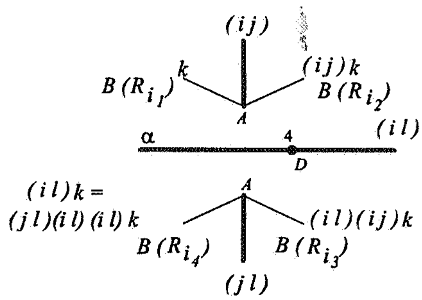


Figure 13b

2. The three arcs have different associated transpositions.

So, the color of one of the $B(R_i)$d etermines the color of the rest. Furthermore, because of the coloration rule, we have that these colors are compatible with ω. (Again, all this is a consequence of the Wirtinger relations that are verified at each crossing point of the diagram.) In this way, we have colored all the vertices of the triangulation Δ.

However, this coloration is not in general a 4-coloration, because we could get two adjacent vertices with the same color. Since the only vertices that are given the number 4 are D-vertices and they are not adjacent, then the only adjacent vertices that could have the same color have to be A or B-vertices. Let us assume that we have a fixed A-vertex that has color k and is an end of an arc α with associated transposition $(i\ j)$. Suppose that it connects to a B-vertex with the same color k. In the Figure 14a we illustrate what happens at the crossing point, at which the fixed A belongs.

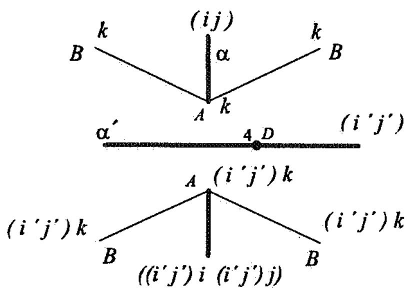


Figure 14a

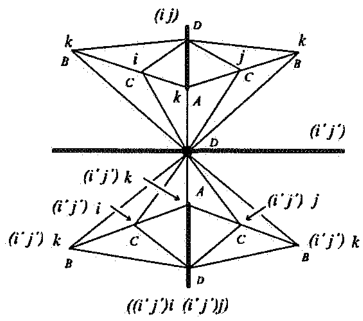


Figure 14b

Actually, there are two B-vertices that connect to the same fixed A-vertex, and both have to have the same color k because the number k is fixed under the action of $(i\ j)$. Now, let α' be the arc that pass over at this crossing point. Suppose that the associated transposition of α' is $(i'\ j')$, then we see that the A and B-vertices placed on the other side of the arc α' have the same color $(i'\ j')k$. Therefore there are four vertices to be adjusted by subdivision. We take a point in the interior of each of these four edges in such a way that all four become identified by the reflections performed along α and α'. We call these points *C-vertices.* We add them to the initial set of vertices of Δ. Now, we trace curves from each of these C-vertices to connect them to the two D-vertices placed in the two adjacent wings to C. We add these new edges to the triangulation Δ. See Figure 14b. To color these four C vertices, we take any of them and color it with $i \in \{1,2,3\} - \{k\}$, if it belongs to the arc whose ends are colored with k. Immediately, because of the Wirtinger relations the colors of the other three are determined and they are compatible with ω. We see this situation in Figure 14b. Moreover, the coloration of the four C points around a crossing point is local, that is, it is independent of the coloration of any other C vertex placed at a different crossing point (if it exists). Since, it is possible to color the four C vertices wherever they appear, we get a 4-colored triangulation compatible with ω. □

Definition 4. *A triangulation obtained by the method given in the last proof is called a canonical 4-colored triangulation.*

Remark 1. *We observe that given a diagram of a knot, it is probable that the number of canonical 4-colored triangulations would be more than three. In fact, once a color is elected for $B(R_1)$ we will have some crossing points (probably different with each of the three elections) where we have to adjoin type C vertices. Furthermore, since we have two choices for coloring the C vertices at each crossing point we will have many options for choosing the coloration.*

Example 5.

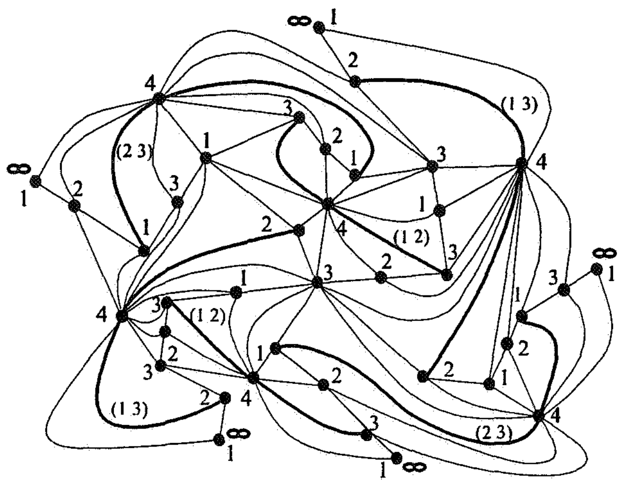


Figure 15

Figure 15 shows a regular diagram for the 2-bridge knot 6_1, (See the knot table in [7].) with a canonical 4-colored triangulation for the boundary of its associated butterfly. For simplicity, we have placed the point at the infinity in several places in the picture, but we note that ∞ represents only one point.

Figures 16a., b. and c. show the crossing points where we have to subdivide to correct coloration discrepances (i.e., where we adjoin C vertices) when we give different colors to ∞. Each figure corresponds to a different canonical triangulation. For example, Figure 15 illustrates one of the 4-colorations that we could obtain for the Figure 16a.

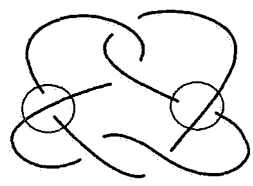


Figure 16a
∞ is colored with 1.

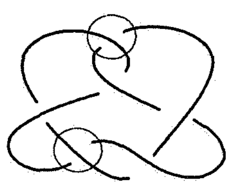


Figure 16b
∞ is colored with 2.

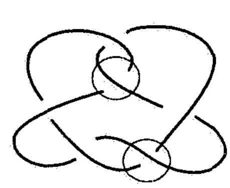


Figure 16c
∞ is colored with 3.

The following Theorem is a natural generalization of the last theorem. It applies to the dihedral representations that occur for the rational knots.

Theorem 4. *Let K be a knot with a given diagram. Let $\omega : \pi_1(S^3 \backslash K) \to \Sigma_{2n+1}$ be a representation that associates to each arc of the diagram a permutation $(i_1, i_2)(i_3, i_4)...(i_{2n-1}, i_{2n})(j)$, where $j \neq i_k$, for $k = 1, ..., 2n$, and all the transpositions are disjoint. Then there is at least a triangulation, of the butterfly boundary, associated to the given diagram that is $2(n+1)$-colored and is compatible with ω.*

Proof. We start with the same triangulation Δ. Similarly to the case before we assign the color $2(n+1)$ to all D-vertices. To color the other vertices, it is enough to observe that the Wirtinger relations are verified at each crossing point and we use them to color the A and B-vertices. If it is neccesary, we have to perform subdivisions on Δ to introduce C-vertices avoiding discrepances and getting a $2(n+1)$ -coloration compatible with ω. □

6. Paths and colorations.

As we said in the introduction, I. Izmestiev and M. Joswig used the group of projectivities $\Pi(\Delta)$, where Δ is a triangulation of a given manifold to construct branched coverings over it. Here we do not use $\Pi(\Delta)$, instead we use the fundamental group of the complement of the codimension 2-skeleton of Δ in the given manifold. In Theorem 5 we show how to construct representations of this group in a symmetric group, in other words with this representation we will show how to construct branched coverings.

In this section, for simplicity, we restrict our attention to dimension 2, eventhough, all the results are also true for dimension 3. Let M be a connected compact closed 2-manifold with a triangulation Δ. In a triangulation Δ, if two different triangles intersect, their intersection is either one vertex or one edge.

Let us consider the fundamental group of the complement of $\Delta_0$in M, $M\backslash\Delta_0$, where Δ_0 is the 0-skeleton of Δ. We take the base point $*$ in the interior of a triangle σ_1 of Δ and we denote $\pi_1(M\backslash\Delta_0,*)$ by $\pi_1(\Delta,\sigma_1)$.

6.1. **Paths of triangles.** We say that two different triangles are *adjacent* if they share one edge, and that a finite sequence $(\sigma_1,...,\sigma_n)$ of triangles of Δ is a *path of triangles* if σ_i and σ_{i+1} are adjacent triangles, for any $i=1,...,n-1$. We say that n is the *length* of $(\sigma_1,...,\sigma_n)$. If $\sigma_n=\sigma_1$, the sequence $(\sigma_1,\sigma_2,...,\sigma_{n-1},\sigma_n=\sigma_1)$ is called a *loop of triangles*. Since $I=[0,1]$ is compact, for any path $\sigma: I\to M\backslash\Delta_0$, there is a unique path of triangles $(\sigma_1,...,\sigma_n)$, such that σ crosses (following the parameter) the triangles in the sequence consecutively. We say that the path of triangles $(\sigma_1,...,\sigma_n)$ is associated to σ.

It is not difficult to see that for any class $[\sigma]$ of homotopic paths in $M\backslash\Delta_0$, we can choose a class representative, such that its associated path of triangles has minimal length. Moreover, this path of triangles is unique. Reciprocally, given a path of triangles $(\sigma_1,...,\sigma_n)$, then each pair of paths $\sigma,\delta: I\to M\backslash\Delta_0$, such that both start at the barycentre of σ_1, both finish at the barycentre of σ_n and both cross consecutively the same sequence $(\sigma_1,...,\sigma_n)$, have to be homotopic. In this sense, the class $[\sigma]$ is denoted by $[\sigma_1,...,\sigma_n]$, where the path $(\sigma_1,...,\sigma_n)$ has the minimal length among all the paths of triangles that are associated to elements in $[\sigma]$. We emphasize that $[\sigma]=[\sigma_1,...,\sigma_n]$ means that n is the minimal length.

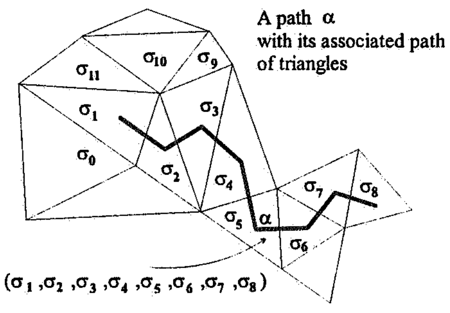


Figure 17
A path with its associated path of triangles.

6.2. **Propagation of colors.** Let σ_1, σ_2 be two adjacent triangles of Δ. Let us assume that σ_1 is 3-colored. There is a natural way to give a 3-coloration to σ_2. In fact, we keep the given color for the two common vertices of σ_1 and σ_2 and we give to the third vertex of σ_2 the same color as the other vertex of σ_1. We say that the color of σ_1 *has been propagated to* σ_2.S ee Figure 18. We notice that a triangulation $\{\sigma_1, \sigma_2\}$, where σ_1 and σ_2 share one edge, become 3-colored.

Nevertheless, if the triangulation has more than two triangles, the propagation of colors does not always produce 3-colorations. For example, a loop $(\sigma_1, \sigma_2, ..., \sigma_n = \sigma_1)$ of triangles could not get a 3-coloration by propagation because if we start giving a 3-coloration to σ_1, and we propagate it through the sequence of triangles, there is no guarantee that the vertices of σ_n, that are equal to σ_1, will get the same colors that we gave at the beginning. Actually, we obtain a permutation of the three colors, that is associated to the giving loop of triangles.

For instance, in Figure 18, if we propagate the given color to σ_0 through the loop $(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_1, \sigma_0)$ we get the permutation (1 2) for the colors of σ_0.

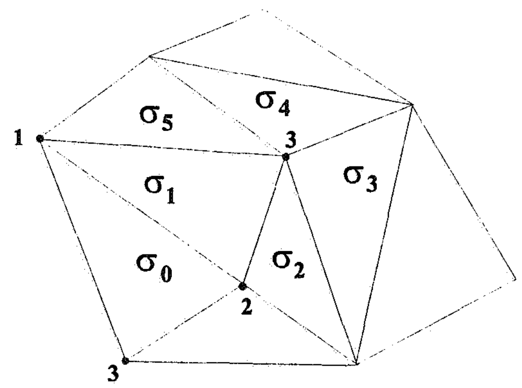


Figure 18

The 3-coloration of σ_0 has been propagated to σ_1.

Since, for each $[\sigma]$ in $\pi_1(\Delta, \sigma_1)$, there is a unique loop of triangles $(\sigma_1, \sigma_2, ...,$ $\sigma_{n-1}, \sigma_n = \sigma_1)$s uch that $[\sigma] = [\sigma_1, \sigma_2, ..., \sigma_{n-1}, \sigma_n = \sigma_1]$ we may define a map $\omega(\Delta) : \pi_1(\Delta, \sigma_1) \to \Sigma_3$, such that if $[\sigma] = [\sigma_1, \sigma_2, ..., \sigma_{n-1}, \sigma_n = \sigma_1] \in \pi_1(\Delta, \sigma_1)$ then $\omega(\Delta)([\sigma])$ is the permutation associated by the last method to the loop $(\sigma_1, \sigma_2, ..., \sigma_{n-1}, \sigma_n = \sigma_1)$. Then it is possible to prove the next theorem.

Theorem 5. *If M^2 is a compact, connected 2-manifold (with or without boundary) with a triangulation Δ, $\sigma_1 \in \Delta$ and if σ_1 is 3-colored, then the map $\omega(\Delta)$: $\pi_1(\Delta, \sigma_1) \to \Sigma_3$ is well defined, and, moreover, it is a representation of the fundamental group $\pi_1(\Delta, \sigma_1)$ in Σ_3, which is canonical (up to conjugation).*

In conclusion, for each triangulation Δ of M^2 there is a canonical representation $\omega(\Delta) : \pi_1(\Delta, \sigma_1) \to \Sigma_3$. We call it *the representation associated to* Δ. We notice that if Δ admits a 3-coloration, then $\omega(\Delta)$ is the trivial representation.

Similarly in higher dimension, if M^3 is a compact connected 3-manifold with a triangulation Δ, there is a canonical representation $\omega(\Delta) : \pi_1(\Delta, \sigma_1) \to \Sigma_4$ associated to Δ. In fact, if we consider $\pi_1(\Delta, \sigma_1) =: \pi_1(M \backslash \Delta_1, *)$, where Δ_1 is the 1-skeleton of Δ, then all the results of this section generalize without difficulty.

Remark 2. *Notice that given a fixed path of triangles that starts at σ_1, then by propagation each vertex of σ_1 has a well defined orbit. Moreover, if the path of triangles is 4-colored the orbit of a fixed vertex of σ_1 has only one color, i.e., there are no permutations of colors inside the orbit.*

For example: Let us consider the loop of tetrahedra that surrounds a fixed edge α, it is easy to see that if the number of different tetrahedra in this loop is even the loop can be 4-colored, otherwise it will induce the permutation $(i\ j)$, where i a nd j are the colors of the vertices not placed on α. In the first case the orbit of each vertex has only one color, and in the second case the orbit of the vertices of α have only one color but the others have two colors: i and j.

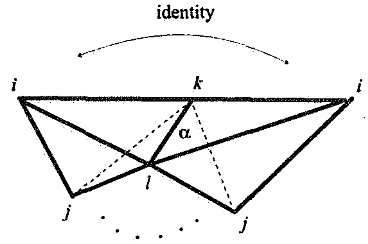


Figure 19a
An even number of tetrahedra.

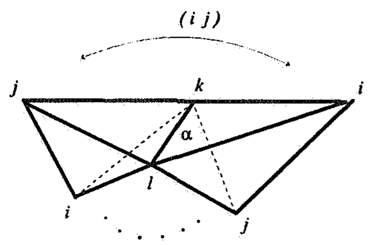


Figure 19b
An odd number of tetrahedra.

Definition 5. *Let Δ be a triangulation in a compact connected 3-manifold M. If α is an edge in Δ, a meridian of α in $\pi_1(M-\Delta,*)$ is a loop σ_α that is the composition of three paths $g\circ m\circ g^{-1}$, where $g:[0,1]\to M-\Delta$ is such that $g(0)=*$, $g(1)$ is a point inside a small tubular neighbourhood N of α, and $m:[0,1]\to M-\Delta$ is such that $m(0)=m(1)=g(1)$ and turns around α only once keeping inside N.*

Notice that if $\sigma_\alpha=[\sigma_1,\sigma_2,...,\sigma_{n-1},\sigma_n=\sigma_1]=g\circ m\circ g^{-1}$ is a meridian of some $\alpha\in\Delta$, then there is one $r\in\{1,2,...,n\}$ such that the tetrahedron σ_r is the first in the sequence that contains α. Without lost of generality we assume that $g(1)$ is inside it. The path of tetrahedra $(\sigma_1,\sigma_2,...,\sigma_r)$ is called the tail of the meridian. Moreover, we see that σ_α is the concatenation of the following three pieces: $[\sigma_1,\sigma_2,...,\sigma_r],[\sigma_{r+1},...,\sigma_k],[\sigma_r,\sigma_{r-1},...,\sigma_n=\sigma_1]=[\sigma_1,\sigma_2,...,\sigma_r]^{-1}$, where all the tetrahedra in the middle piece contain α. The loop of tetrahedra $(\sigma_r,\sigma_{r+1},...,\sigma_k,\sigma_{k+1}=\sigma_r)$ will be called the real meridian. Notice that $k=n-r$.

7. The Izmestiev-Joswig Theorem

Here we will find the Izmestiev-Joswig Theorem as a corollary of our Theorem 6. First of all, we start this section defining a special kind of subdivision, since in Theorem 6 we will need to subdivide tetrahedra without spoiling the coloration,

7.1. **Antipyramidal subdivision.** Let $v_1v_2v_3$ be a triangle in a triangulation Δ of a 2-manifold M^2. We suppose that the colors of the vertices are the same subindices. Let B be its barycentre. First we construct its barycentric subdivision. Let B_1, B_2 and B_3 be the barycentres of the triangles Bv_2v_3, Bv_1v_3 and Bv_1v_2, respectively, that we color with their subindices. The *antipyramidal subdivision* has the following 7 triangles $B_1B_2B_3$, $v_1v_2B_3, v_1v_3B_2, v_2v_3B_1, v_1B_2B_3, v_2B_1B_3$ y $v_3B_1B_2$. We extend the coloration of the initial triangle to this new triangulation.

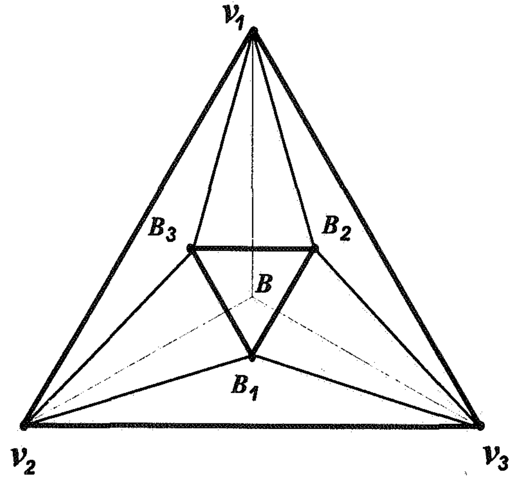


Figure 20

Similarly, for dimension 3, if $v_1v_2v_3v_4$is a tetrahedron with barycentre B, its antipyramidal subdivision contains the following 13 tetrahedra $B_1B_2B_3B_4$, $v_1v_2v_3B_4$, $v_1v_2v_4B_3$, $v_1v_3v_4B_2$, $v_2v_3v_4B_1$, $v_1v_2B_3B_4$, $v_1v_3B_2B_4$, $v_2v_3B_1B_4$, $v_3v_4B_1B_2$, $v_1B_2B_3$ B_4, $v_2B_1B_3B_4$, $v_3B_1B_2B_4$ and $v_4B_1B_2B_3$, where B_i is the barycentre of $Bv_jv_kv_l$, with j,k,l different numbers in $\{1,2,3,4\}$. As in the 2 dimension case, this subdivision has the advantage that the coloration of the original tetrahedron extends to it, i.e., it does not spoil it.

Theorem 6. *Given a representation $\omega_0 : \pi_1(S^3\backslash K) \to \Sigma_3$, that sends each arc of a diagram D_K (of a knot K) to a transposition, there exists a triangulation Δ of S^3 whose associated representation $\omega(\Delta) : \pi_1(S^3\backslash\Delta_1) = \pi_1(\Delta,\sigma_1) \to \Sigma_4$ is the natural extension of ω_0, i.e., $\omega(\Delta) = i \circ \omega_0 \circ j$, where $i : \Sigma_3 \hookrightarrow \Sigma_4$ and $j : \pi_1(S^3\backslash\Delta_1) \hookrightarrow \pi_1(S^3\backslash K)$ are the canonical inclusions.*

Proof. Let (D^*_K, R, T) be the butterfly associated to the diagram D_K. By Theorem 3, there exists a triangulation Δ^2o n the boundary S of the ball D^*_K, that admits a 4-coloration and is compatible with ω_0. Observe that the number 4 is fixed by $i \circ \omega_0$.

Now, we extend this triangulation to the butterfly interior by using a constructive theorem of Goodman and Onishi that says that given a 4-colored triangulation of S^2 it is possible to extend it to a 4-colored triangulation of the 3-ball B^3 (see [2]).

Once we have the triangulation of D^*_K, identifying the butterfly wings we obtain a triangulation Δ of S^3. However, we have to be careful! The obtained triangulation might not be a good triangulation, since we could get two tetrahedra sharing more than one face or one tetrahedron with two of its faces identified between them. If this occurs, it is enough to make an antipyramidal subdivision to each of the problematical tetrahedra.

Anyway, the obtained triangulation Δ of S^3 is such that $\omega(\Delta) = i \circ \omega_0 \circ j$. In fact, it is enough to see what happens to the meridians (see Definition 5) of all edges of Δ, since they are generators of $\pi_1(S^3\backslash\Delta_1)$. Let α be an edge in Δ and let $[\sigma_\alpha] = [\sigma_1, \sigma_2, ..., \sigma_{n-1}, \sigma_n = \sigma_1] = [\sigma_1, \sigma_2, ..., \sigma_r][\sigma_{r+1}, ..., \sigma_k][\sigma_1, \sigma_2, ..., \sigma_r]^{-1}$ be a meridian of α, where $(\sigma_1, \sigma_2, ..., \sigma_r)$ is its tail. Without lost of generality, we could assume that no common face in the consecutive tetrahedra in the tail is contained in the boundary S. So the tail is in D_K^* and can be 4 colored. It implies that if we choose a coloration for σ_1th en σ_r get colored and vice versa. So, in order to see what is $\omega(\Delta)([\sigma_\alpha])$ it is enough to see what happens to the colors of σ_r when we propagate them through the real meridian $(\sigma_r, \sigma_{r+1}, ..., \sigma_k, \sigma_{k+1} = \sigma_r)$.

We also observe that if $\alpha \notin p(T)$, where $p : (D_K^*, R, T) \to S^3$ is the natural projection, then $j([\sigma_\alpha])$ is nullhomotopic in $\pi_1(S^3 \backslash K)$, so $i \circ \omega_0 \circ j([\sigma_\alpha])$ has to be the identity. Now if $\alpha \in p(T)$, then $[\sigma_\alpha]$ is a meridian of an arc of K, and so $i \circ \omega_0 \circ j([\sigma_\alpha]) = i \circ \omega_0([\sigma_\alpha])$ is equal to the transposition associated to the arc of K that contains the edge α.

We have to consider four cases:

a) Suppose the edge α belongs to $p(T)$. In this case there exists exactly one $i \in \{r, r+1, ..., k\}$ such that the common face between σ_i and σ_{i+1} has two copies on S. So the real meridian is formed by the concatenation of two paths of tetrahedra $(\sigma_r, \sigma_{r+1}, ..., \sigma_i)$ and $(\sigma_{i+1}, ..., \sigma_{k+1} = \sigma_r)$, that are 4-colored because each one is contained in D_K^*.A ssume that the α-reflection induces the permutation $(j\ k) \in \Sigma_4$, then we see that if v_1 is, for example, the vertex of σ_r with color $l \in \{1, 2, 3, 4\}$, then the orbit (see Remark 2) of this vertex has the color l, while it is inside the path of tetrahedra $(\sigma_r, \sigma_{r+1}, ..., \sigma_i)$ and change to the color $(j\ k)l$ once it is inside the second piece $(\sigma_{i+1}, ..., \sigma_k = \sigma_r)$. So for this case $\omega(\Delta)([\sigma_\alpha]) = i \circ \omega_0 \circ j([\sigma_\alpha]) = (j\ k) \in \Sigma_4$

b) Suppose the edge α is in the interior of D_K^* (i.e., $\alpha \notin p(T)$). Then for every $i = r, r+1, ..., k$, the common face between σ_i and σ_{i+1} does not belong to the butterfly boundary so the real meridian is contained in D_K^*, it is 4-colored, and therefore $\omega(\Delta)([\sigma_\alpha])$ is the identity in Σ_4. (See Remark 2.)

c) Suppose $\alpha \notin p(T)$ and $p^{-1}(\alpha)$ is a set of 2 edges of $S-T$ that are placed inside a pair of wings $A, \bar{A}$. In this case there exist exactly two numbers i, s in $\{r, r+1, ..., k\}$ $(i < s)$, such that the common face between σ_i and σ_{i+1} (resp. σ_s and σ_{s+1}) has two copies on S. Therefore the real meridian is formed by the concatenation of three pieces $(\sigma_r, \sigma_{r+1}, ..., \sigma_i)$, $(\sigma_{i+1}, ..., \sigma_s)$ and $(\sigma_{s+1}, ..., \sigma_{k+1} = \sigma_r)$. Each of them is contained in D_K^* and is 4-colored. Assume that the reflection, that identifies the wing A with the wing $\bar{A}$ induces the permutation $(j\ k) \in \Sigma_4$. Then we see that if v_1 is, for example, the vertex of σ_r with color $l \in \{1, 2, 3, 4\}$, then the orbit (see Remark 2) of this vertex has the color l, while it is inside the path of tetrahedra $(\sigma_r, \sigma_{r+1}, ..., \sigma_i)$, changes to the color $(j\ k)l$ once it is inside the second piece $(\sigma_{i+1}, ..., \sigma_s)$ and changes again to the color $(j\ k)(j\ k)l = l$ once it is inside the third piece $(\sigma_{s+1}, ..., \sigma_k, \sigma_{k+1} = \sigma_r)$. So for this case $\omega(\Delta)([\sigma_\alpha])$ is also the identity in Σ_4.

d) Suppose $\alpha \notin p(T)$ and $p^{-1}(\alpha) = \{\alpha_i\}_{i=1}^t$ is a set of t edges on $S - T$, each of them placed on the boundary of some wing.In this case the real meridian is formed by the concatenation of $t+1$ pieces $[\sigma_r, \sigma_{r+1}, ..., \sigma_{i_1}]$, $[\sigma_{i_1+1}, ..., \sigma_{i_2}]$, $[\sigma_{i_2+1}, ..., \sigma_{i_3}]$, ..., $[\sigma_{i_s+1}, ..., \sigma_{i_{s+1}}]$, ..., $[\sigma_{i_{t-1}+1}, ..., \sigma_{i_t}]$, and $[\sigma_{i_t+1}, ..., \sigma_k, \sigma_{k+1} = \sigma_r]$, where each of them is contained in D_K^*; all the tetrahedra in the piece $(\sigma_{i_l+1}, ..., \sigma_{i_{l+1}})$ contain the edge α_{l+1} (for any $l \in \{1, 2, ..., t\}$), and the common face between σ_{i_l} and σ_{i_l+1}, that we will denote by $< \sigma_{i_l}, \sigma_{i_l+1} >$, for any $l \in \{1, ..., t\}$, has two copies placed on the boundary S. Since α does not belong to $p(T)$, we see (by the construction of the triangulation Δ^2, given in Theorem 3) that the vertex of both of the copies of $< \sigma_{i_l}, \sigma_{i_l+1} >$ (for any $l \in \{1, ..., t\}$) that is not in $p^{-1}(\alpha)$is a D-vertex that has to be colored with the number 4. Moreover, since all the paths of tetrahedra $(\sigma_{i_1+1}, ..., \sigma_{i_2})$, $(\sigma_{i_2+1}, ..., \sigma_{i_3})$, ..., $(\sigma_{i_{r-1}+1}, ..., \sigma_{i_r})$ are 4-colored and the number 4 is fixed by $i \circ \omega_0$, we have that each of these paths has to have an even number of tetrahedra (see Remark 2). The next figure shows one of these pieces.

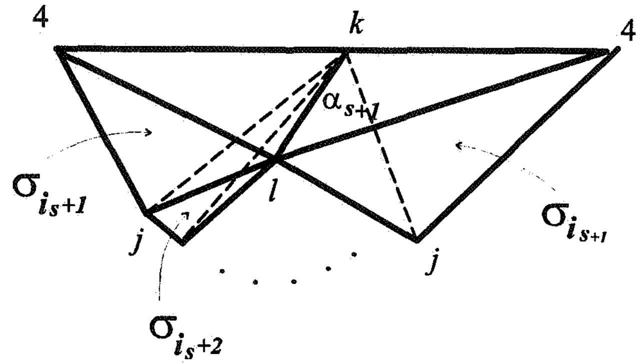


Figure 21

But also the path formed by the concatenation of $(\sigma_r, \sigma_{r+1}, ..., \sigma_{i_1-1}, \sigma_{i_1})^{-1}$w ith $(\sigma_{i_t+1}, \sigma_{i_t+2}, ..., \sigma_{k-1}, \sigma_k)^{-1}$, that is $(\sigma_{i_1}, \sigma_{i_1-1}, ..., \sigma_{r+1}, \sigma_r = \sigma_{k+1}, \sigma_{k,\kappa-1}, ..., \sigma_{i_t+2}, \sigma_{i_t+1})$ is such that all the common faces between consecutive tetrahedra in the sequence do not belong to S. Therefore, the path is also in D_K^*, is 4 colored and also has the form of the Figure 21. So summing up the number of tetrahedra around α, we get an even number. Then, again $\omega(\Delta)([\sigma_\alpha])$ is the identity in Σ_4.

Therefore, in any case we have proved that $\omega(\Delta)([\sigma_\alpha]) = i \circ \omega_0 \circ j([\sigma_\alpha])$. □

Here we have the Izmestiev-Joswig Theorem:

Corollary 1. *For each closed orientable 3-manifold M there is a triangulation Δ of S^3, such that if $\rho(\omega(\Delta)) : \check{N} \to N$ is the branched covering associated to the canonical representation $\omega(\Delta) : \pi_1(S^3 \backslash \Delta_1) \to \Sigma_4$, then M is one of the components of $\check{N}$ and the restriction $\rho(\omega(\Delta))/M$ is a simple 3-covering branched over a knot that is contained in the 1-skeleton $\Delta_1 o\ f\ \Delta$.*

Proof. In fact, it is known that for any 3-manifold M there is a simple 3-covering branched over a knot (see [3] and [8]). In other words, for each 3-manifold there is a knot K and one representation $\omega_0 : \pi_1(S^3 \backslash K) \to \Sigma_3$ that sends each arc of a diagram D_K of K to a transposition in Σ_3. Therefore the hypothesis of the last theorem is verified and so the existence of the triangulation of S^3 is guaranteed. Furthermore, if $\rho(\omega_0) : \check{S} \to S^3$ is the branched covering associated to the representation ω_0, then $\check{S} = M$. On the other hand, since $\rho(\omega_0) : M \to S^3$ and $\rho(\omega(\Delta)) : \check{N} \to S^3$ are the branched coverings associated to ω_0 and to $\omega(\Delta) = \omega$, respectively, it follows that M is the component of $\check{N}$ such that $\rho(\omega(\Delta))/M$ is $\rho(\omega_0)$. □

Example 6.

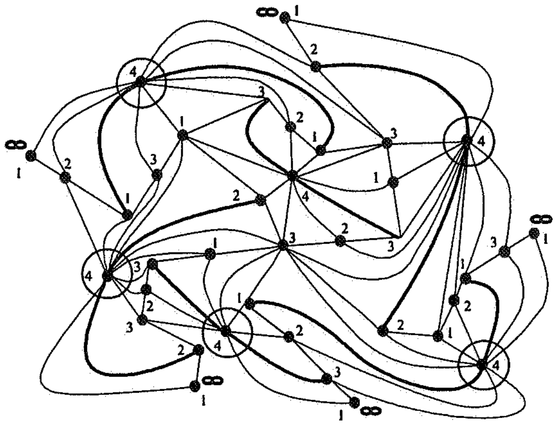


Figure 22

Now, as an example, we are going to construct the triangulation of S^3 described in the last corollary for the knot 6_1. To obtain it, we follow the constructive procedures described in [2]. In the last picture, we have a 4-colored triangulation for the boundary butterfly associated to the knot 6_1.

In order to get the desired triangulation on S^3, we start cutting the five pyramids whose vertices are circled in the last picture of the butterfly. We recall that the points labeled by ∞ represent the point at the infinity. We obtain a 3-ball B (contained in the butterfly) whose boundary is poligonalized (or celled) once we cut out these pyramids. Its boundary is shown in the following picture.

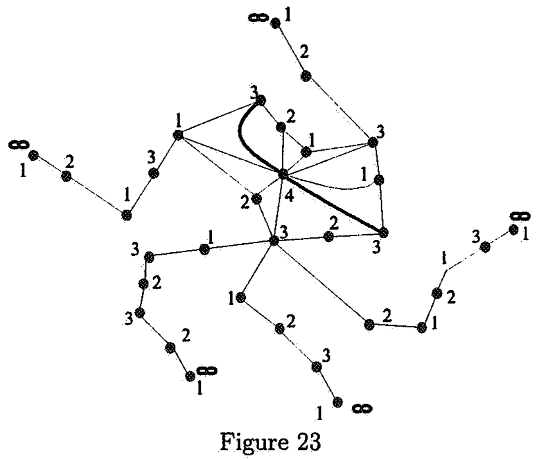


Figure 23

Next, we triangulate the base of each of the pyramids, being careful with the coloration. We perform these triangulations following [2]. In this way, we obtain a 4-colored triangulation of the boundary of B. It has only one vertex with assigned color 4. Joining this vertex to all others, using interior edges in B we get a 4-colored triangulation of B. We denote it by $\Delta(B)$. On the other hand we triangulate each of the 5 pyramids in a natural way; i.e., we subdivide each pyramid in order to get as many tetrahedra as triangles in the base (again by [2] we can choose the triangles in the base in such a way that each triangle has 3 different colors on its vertices). Each of these tetrahedron has as vertex the same vertex of the respective pyramid, which is circled in Figure 22 and has color 4. The triangulations of the pyramids are denoted by $\Delta(P_i)$, $i = 1, ..., 5$. Thus a 4-colored triangulation for the complete butterfly is $\Delta(B) \cup (\cup_{i=1}^{5} \Delta(P_i))$. We observe that we have added neither vertices, edges nor faces to the butterfly boundary. All the additions were performed inside the butterfly.

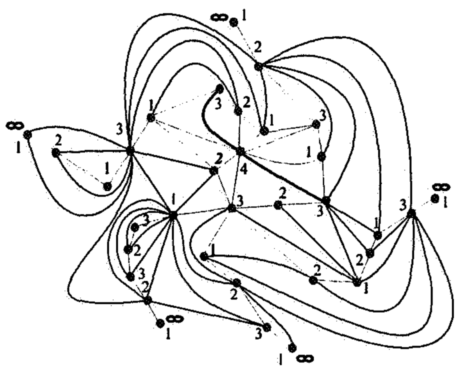


Figure 24

Now, to get S^3 from the butterfly, we have to identify the butterfly wings but also we have to be sure that we get a good triangulation, i.e., two different tetrahedra can have at most one common face and two different faces can not be identified if they belong to the same tetrahedron. In order to get a good triangulation we do antypyramidal subdividivision wherever it is necessary. To be sure we subdivide all tetrahedra in each $\Delta(P_i)$ $(i = 1, ..., 5)$, while in $\Delta(B)$ we subdivide each of the tetrahedra that have a face that is marked with a square in the following picture. These marked faces are just those on the butterfly boundary. We notice that, perhaps, we are subdividing more tetrahedra than necessary.

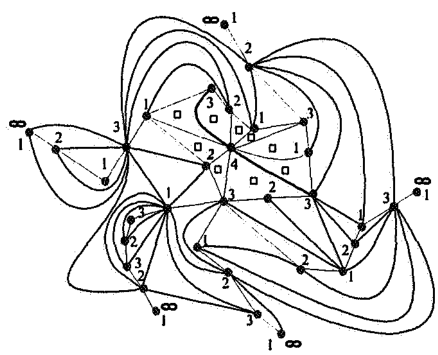


Figure 25

Finally, we identify the butterfly wings and we obtain the desired triangulation for S^3.

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