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32

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DISTINGUISHED FRÉCHET SPACES, DUAL METRIC SPACES AND TIGHTNESS CONDITIONS FOR $C_C(X)$

DISTINGUISHED FRÉCHET SPACES DUAL METRIC SPACES AND TIGHTNESS CONDITIONS FOR $C_c(X)$

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ABSTRACT. The aim of this note is to gather some of recent results presented in my talk September 22, 2004 in Departamento de Mathemáticas de la U.N.E.D in Madrid. I would like to thank Professor Pedro Jemenez Guerra for his invitation to give this talk and a stimulating discussion after the talk. The talk entitled as above covered some of recent results jointly obtained with my colleagues B. Cascales (Murcia), J. C. Ferrando (Elche), M. Lopez-Pellicer (Valencia), S. Saxon (Gainesville) and A. Tood (New York). The main idea of my talk was to present some technics and ideas from descriptive set topology related to K-analytic spaces, analytic spaces and quasi-Suslin spaces, to study concrete problems from functional analysis like characterizations of distinguished Fréchet spaces, dual metric spaces and tightness conditions for function spaces $C_c(X)$. Our approach provides also a new description of distinguished Fréchet spaces in terms of tightness and dominating ordinals. Several results of Bierstedt-Bonet, Kaplansky, Talagrand, Morris-Wulbert are essentially extended.

Recall that a locally convex space (lcs) is called a *dual metric space* if it has a fundamental sequence of bounded sets and every strongly bounded sequence in the dual is equicontinuous. Clearly every Grothendieck (DF)-space is dual metric and every dual metric space belongs to class $\mathfrak{G}$.

In [4] Cascales and Orihuela introduced the class $\mathfrak{G}$ of those lcs E for which there is a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of its topological dual E' (called its $\mathfrak{G}$-representation) such that:

(a) $E' = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$;

(b) $A_\alpha \subset A_\beta$ when $\alpha \leqslant \beta$;

(c) in each A_α, sequences are equicontinuous,

where the set of natural numbers $\mathbb{N}$ is endowed with the discrete topology and $\mathbb{N}^{\mathbb{N}}$ with its product topology.

Cascales and Orihuela obtained the following two important results:

(1) *Every lcs E in class $\mathfrak{G}$ is angelic, both in the original and the weak topologies.*

(2) *Every compact set in E is metrizable.*

Last result (2) answers a question of Floret [13].

Note that condition (c) implies that every set A_α is bounded in the strong topology $\beta(E',E)$ of E'. Moreover every A_α is countably $\sigma(E',E)$-relatively compact. Also each A_α is *bounding*, i.e. every continuous real-valued map on the weak dual $(E',\sigma(E',E))$ is bounded on A_α. This facts have some nice applications. For instance, if the weak dual $(E',\sigma(E',E))$ is *realcompact*, i.e. $(E',\sigma(E',E))$ is homeomorphic to a closed subspace of $\mathbb{R}^I$ for some I, then every bounding set in $(E',\sigma(E',E))$ must be relatively compact. Therefore every set A_α is relatively compact; consequently the weak dual $(E',\sigma(E',E))$ is covered by an ordered family of compact sets. Nevertheless, this property does not ensure that $(E',\sigma(E',E))$ is a K-analytic space. Our Theorem 3 below describes relations between tightness of the weak topology of a lcs E in class $\mathfrak{G}$ and several properties of the weak dual of E like Lindelöf or K-analytic properties.

EXAMPLES OF SPACES IN CLASS $\mathfrak{G}$

All (LM)-spaces (hence also all (LF)-spaces), dual metric spaces (hence (DF)-spaces), the spaces of distributions $D'(\Omega)$ and real analytic functions $A(\Omega)$ for open $\Omega \subset \mathbb{R}^{\mathbb{N}}$, etc., belong to class $\mathfrak{G}$, see [3], [8].

The class $\mathfrak{G}$ is stable by taking subspaces, separated quotients, completions, countable direct sums and countable products, [4]. The following results show the importance of quasibarrelled spaces in class $\mathfrak{G}$. Note that "almost all" important spaces in class $\mathfrak{G}$ are quasibarrelled, although Grothendieck (DF)-spaces need not to be quasibarrelled.

Theorem 1 [8]. *Let E be a quasibarrelled lcs. The following assertions are equivalent:*

(1) $E \in \mathfrak{G}$.

(2) The strong dual $(E',\beta(E',E))$ is a quasi-(LB)-space.

(3) E admits a $\mathfrak{G}$-basis, i.e. a basis of neighbourhoods of zero $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in E such that $U_\alpha \subset U_\beta$ for all $\beta \leq \alpha$ in $\mathbb{N}^{\mathbb{N}}$.

(4) For every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ there exists a family of absolutely convex closed subsets

$$\mathcal{F} := \{D_{n_1,n_2,\dots,n_k} : k, n_1, n_2, \dots, n_k \in N\}$$

of X such that

(a) $D_{n_1,n_2,\dots,n_k} \subset D_{m_1,m_2,\dots,m_k}$, if $n_i \leqslant m_i$ for $i = 1, 2, \dots, k$.

(b) For every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ we have $D_{n_1} \subset D_{n_1,n_2} \subset \cdots \subset D_{n_1,n_2,\dots,n_k}$ and the sequence is bornivorous.

(c) If $W_\alpha := \bigcup_k D_{n_1,n_2,\ldots,n_k}$, where $\alpha \in \mathbb{N}^{\mathbb{N}}$, then the family $\{W_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighbourhoods of zero in E.

In a series of papers [7], [8], [10], [11] we studied some tightness conditions for spaces in $\mathcal{G}$ and those spaces in class $\mathcal{G}$ which have countable tightness when endowed with their weak topology. The following two results extend a classical Kaplansky result and provide many applications.

THEOREM 2 [7]. *(1) Every quasibarrelled space in class $\mathfrak{G}$ has countable tightness. Hence all (LM)-spaces, dual metric spaces (hence (DF)-spaces), the spaces of distributions $D'(\Omega)$ and real analytic functions $A(\Omega)$ for open $\Omega \subset \mathbb{R}^{\mathbb{N}}$ have countable tightness.*
(2) If a lcs E in class $\mathfrak{G}$ has countable tightness, then the space E endowed with its weak topology $\sigma(E, E')$ has also countable tightness.

Recall that a topological space X has *countable tightness* if for every set A in X and every $x \in \overline{A}$ (closure in X) there exists a countable subset B of A whose closure contains x.

THEOREM 3 [7]. *For a lcs E in class $\mathfrak{G}$ the following conditions are equivalent:*
(a) $(E, \sigma(E, E'))$ has countable tightness.
(b) $(E', \sigma(E', E))$ is K-analytic.
(c) $(E', \sigma(E', E))$ is realcompact.
(d) $(E', \sigma(E', E))$ is Lindelöf.
(e) $(E', \sigma(E', E))^n$ is Lindelöf for every $n \in \mathbb{N}$.

On the other hand the following very interesting applicable result (Arkhangelski-Pytkeev-McCoy [1], [16]) provides a nice link between tightness of $C_p(X)$ and Lindelöf property of X.

(+) Tightness of $C_p(X)$ is countable iff every finite product of X is Lindelöf.

Note that Theorem 3 does not apply for spaces $C_p(X)$ of continuous functions on X with the pointwice topology. This follows from the following two results of Cascales-Kąkol-Saxon [8] and of Kąkol-Lopez-Pellicer-Todd [15] combined with (+):
(1) *The space $C_p(X)$ belongs to class $\mathfrak{G}$ iff X is countable.*
(2) *If X is an uncountable K-analytic space, then the weak dual of $C_p(X)$ is not a K-analytic space.*

The following fact due to Cascales-Orihuela [5] supplements Theorem 3: *A barrelled space E belongs to class $\mathfrak{G}$ iff its weak dual is K-analytic.* This combined with Theorem 1 yields the following interesting consequence.

COROLLARY 1. *A barrelled space E has a $\mathfrak{G}$-basis iff its weak dual is K-analytic.*

Recall that a Hausdorff topological space X is a *quasi-Suslin* space (resp. *K-analytic*), if there exists a map $T : \mathbb{N}^{\mathbb{N}} \to 2^X$ (resp. a map $T : \mathbb{N}^{\mathbb{N}} \to 2^X$ such that

$T(\alpha)$ is compact for each $\alpha \in \mathbb{N}^{\mathbb{N}}$) such that

(a) $\bigcup\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\} = X$.

(b) If α_n is a sequence in $\mathbb{N}^{\mathbb{N}}$ which converges to α in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$, then the sequence $(x_n)_n$ has an adherent point in X belonging to $T(\alpha)$;

Rogers [20] proved that K-analytic spaces and K-Suslin spaces (in sense of [23]) coincide in the category of completely regular Hausdorff spaces. It is easy to see that a regular Hausdorff space X is K-analytic iff it is quasi-Suslin and Lindelöf. For every K-analytic space X there exists an ordered family $\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact subsets of X covering X, see Talagrand [21] but, as Talagrand has shown [22], there are topological spaces not K-analytic, but covered by an ordered family $\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets.

We note another interesting result which provides also a nice application of (+).

Every separable lcs $\in \mathfrak{G}$ *has countable tightness in* $\sigma(E, E')$.

Proof. If D is a countable dense subset of E, then $\sigma(E', D)$ is metrizable, hence *angelic*. Therefore $(E', \sigma(E', E))$ is angelic, too. Since E belongs to class $\mathfrak{G}$, every set A_α from its $\mathfrak{G}$-representation is countably $\sigma(E', E)$-relatively compact. Since $(E', \sigma(E', E))$ is angelic, so each set A_α is relatively compact. But this applies to deduce that $(E', \sigma(E', E))$ is K-analytic. One of the interesting properties of K-analytic spaces shows that countable products of K-analytic spaces are K-analytic, so each such product is Lindelöf. Hence every finite product $((E', \sigma(E', E)))^n$ must be Lindelöf for any $n \in \mathbb{N}$. Now (+) applies to show that $C_p(E', \sigma(E', E))$ has countable tightness. Since $(E, \sigma(E, E'))$ is a subspace of $C_p(E', \sigma(E', E))$, the conclusion holds.

$\mathfrak{G}$-CLOSED AND BORNIVOROUS REPRESENTATIONS

A $\mathfrak{G}$-representation $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of a lcs E is *closed* if every set A_α is $\sigma(E', E)$-closed. It is called *bornivorous* if every bounded set in the strong topology $\beta(E', E)$ of E' is contained in some A_α.

It turns out that this subclass of spaces in class $\mathfrak{G}$ is still very large as examples below show.

EXAMPLE. *The following lcs admit a closed and bornivorous* $\mathfrak{G}$*-representation*:

(A) *Every dual metric space*; consequently every (DF)-space and every (LB)-space.

(B) *Every quasibarrelled space in class* $\mathfrak{G}$; hence every (LM)-space.

(C) $D'(\Omega)$, $A(\Omega)$: For open sets $\Omega \subset \mathbb{R}^{\mathbb{N}}$, the space of test functions $D(\Omega)$ is a complete Montel (LF)-space, so its strong dual, the space of distributions $D'(\Omega)$, is a quasi-complete ultrabornological (hence quasibarrelled) and non-metrizable lcs. The same holds for $A(\Omega)$. The spaces $D'(\Omega)$ and $A(\Omega)$ belong to class $\mathfrak{G}$, so both spaces admit a bornivorous $\mathfrak{G}$-representation and have $\mathfrak{G}$-bases.

The following theorem extends Valdivia's Theorem 24 of [23] for lcs with bornivorous $\mathfrak{G}$-representation.

THEOREM 4 [11]. *Let E be a lcs with a bornivorous $\mathfrak{G}$-representation (for example a dual metric space).*

(I) The following are equivalent:

E has countable tightness.

E is quasibarrelled.

(II) The following are equivalent:

$(E, \sigma(E, E'))$ has countable tightness.

$(E, \mu(E, E'))$ has countable tightness.

$(E, \mu(E, E'))$ is quasibarrelled.

Theorem 4 can be applied to get an interesting new characterization of distinguished metrizable complete lcs (= Fréchet spaces). Recall that a Fréchet space E is *distinguished* if its strong dual $(E', \beta(E', E))$ is *bornological*, or equivalently, *barrelled*. If $(E', \beta(E', E))$ is metrizable, then clearly E is distinguished. But then E is a Banach space. Therefore the following natural question arises:

Could purely topological weakenings of metrizability, such as the countable tightness succeed here ?

Note that the Fréchet-Urysohn property is also to strong: Indeed, if $E = \mathbb{R}^{\mathbb{N}}$, then its strong dual is φ, i.e. the $\aleph_0$-dimensional vector space endowed with the finest locally convex topology, which is not Fréchet-Urysohn but bornological, so E is distinguished. Recall that a topological space X is *Fréchet-Urysohn* if for every set A in X and every point x from the closure of A there exists a sequence of elements of the set A which converges to x, see Nikyos [16] for details. Theorem 4 applies to get the following

COROLLARY 2. *A Fréchet space E is distinguished iff its strong dual has countable tightness.*

This interesting observation will be extended later. In order to get another results related to distinguished spaces in terms of tightness we need the following few concepts.

For $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\alpha = (a_k)_k$ and $\beta = (b_k)_k$ we write $\alpha \leq^* \beta$ to mean that $a_k \leq b_k$ for almost all (i.e., all but finitely many) $k \in \mathbb{N}$.

Thus $\alpha \leq \beta$ implies $\alpha \leq^* \beta$, but not conversely. It is easily seen that every countable set in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ has an upper bound, and this is not true for $(\mathbb{N}^{\mathbb{N}}, \leq)$.

The *bounding cardinal* $\mathfrak{b}$ and the *dominating cardinal* $\mathfrak{d}$ are defined as the least cardinality for unbounded, respectively, cofinal subsets of the quasi-ordered space $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. The *cofinality* of an infinite cardinal $\mathfrak{k}$ is the smallest cardinality for cofinal subsets of $\mathfrak{k}$, where $S \subset \mathfrak{k}$ is *cofinal* if, for each ordinal $\alpha < \mathfrak{k}$, i.e. for each $\alpha \in \mathfrak{k}$, there exists $\beta \in S$ such that $\alpha \leq \beta$. It is clear that in any ZFC-consistent system, one has

$$\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}.$$

The Continuum Hypothesis requires all four of these cardinals to coincide. Yet it is ZFC-consistent to assume that any of the three inequalities is strict.

Now we are ready to present some new facts about famous Grothendieck-Köthe distinguished space.

EXAMPLE. *Grothendieck-Köthe* non-distinguished Fréchet space is the vector space E of all numerical double sequences $x = (x_{ij})$ such that for each $n \in \mathbb{N}$ one has

$$p_n(x) := \sum_{ij} |a_{ij}^{(n)} x_{i,j}| < \infty,$$

where $a_{ij}^{(n)} = j$ for $i \leq n$ and all j, $a_{ij}^{(n)} = 1$ for $i > n$ and all j. The semi-norms p_n generate a locally convex topology under which E is a Fréchet space. The dual E' of E is identified with the space of double sequence $u = (u_{ij})$ such that $|u_{ij}| \leq c a_{ij}^{(n)}$ for all $i, j \in \mathbb{N}$ and suitable $c > 0$ and $n \in \mathbb{N}$.

THEOREM 5 [11]. *The tightness of the strong dual E' of the Grothendieck-Köthe Fréchet space E is $\mathfrak{d}$, the dominating cardinal. Moreover the tightness of the space $(E', \sigma(E', E''))$ is between cardinals $\mathfrak{b}$ and $\mathfrak{d}$.*

This yields the following interesting

COROLLARY 3. *The Grothendieck-Köthe space is, indeed, a non-distinguished Fréchet space.*

We provide another characterization of distinguished Fréchet spaces in terms of tightness. Let E be a lcs and $\mathfrak{U}(E)$ the basis of all closed absolutely convex neighbourhoods of zero. Let $\mathfrak{B}(E)$ be the family of all closed absolutely convex bounded subsets of E.

A classical result of Grothendieck says that *every (DF)-space for which every bounded set is metrizable is quasibarrelled*. Hence a metrizable lcs whose strong dual has all bounded sets metrizable must be *distinguished*. Following S. Heinrich, we say that E satisfies the *density condition*, if the following holds:

Given any function $\lambda : \mathfrak{U}(E) \to \mathbb{R}_+ \setminus \{0\}$ and an arbitrary $V \in \mathfrak{U}(E)$, there exists a finite subset $\mathfrak{U}$ of $\mathfrak{U}(E)$ and $B \in \mathfrak{B}(E)$ such that

$$\bigcap_{U \in \mathfrak{U}} \lambda(U) U \subset B + V.$$

In [3] Bierstedt and Bonet studied the Stefan Heinrich's density condition for metrizable lcs. They noticed that for a metrizable lcs E with its decreasing basis

$(U_n)_n$ of absolutely convex closed neighbourhoods of zero the density condition is satisfied iff the following holds:

Given an increasing sequence $(\lambda_n)_n$ of strictly positive numbers, there exists a bounded subset B of E such that, for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$, $m > n$, and $M > 0$ with $\bigcap_{j=1}^{m} \lambda_j U_j \subset MB + U_n$.

In the same paper [3] Bierstedt and Bonet proved:

THEOREM 6. *For a metrizable lcs E the following assertions are equivalent:*

(1) E satisfies the density condition.

(2) Every bounded set in the strong dual $(E', \beta(E', E))$ is metrizable.

(3) The space $\ell^1(E)$ is distinguished.

Any condition given above implies that E is distinguished; the converse fails in general. There exist reflexive Fréchet spaces E (every such space is clearly distinguished) but which do not satisfy the density condition. Nevertheless, it turns out that for echelon Köthe spaces the density condition characterizes the distinguished property. Indeed, Bierstedt and Bonet proved that:

For an echelon Köthe space $\lambda_1 = \lambda(I, A)$ the following assertions are equivalent:

(1) λ_1 is distinguished.

(2) λ_1 satisfies the density condition,

where I is an arbitrary index set and $A := (a_n)_{n \in \mathbb{N}}$ is a strictly positive Köthe matrix on I.

Very recently we have shown that

THEOREM 7. *A (DF)-space $E = (E, \tau)$ is quasibarrelled iff every bounded set in E has countable tightness.*

Skech of the proof. Let (S_n) be a fundamental sequence of absolutely convex closed sets in E. If E is quasibarrelled, then by Theorem 2 the space E has countable tightness and the conclusion follows. Now we prove the converse. Assume that every bounded set in E has countable tightness. We need the following:

CLAIM 1. *The weak dual $(E', \sigma(E', E))$ is realcompact.*

In order to prove Claim 1 it is enough to show that every linear functional f on $(E, \sigma(E, E'))$ whose restrictions to every separable τ-closed subspace of E are continuous itself is continuous. But to show that the map f is continuous, it is enough to check that for every $n \in \mathbb{N}$ the restriction $f|S_n$ is continuous. Indeed, since the space E is assumed to be a (DF)-space, this will show that f is continuous. Fix $n \in \mathbb{N}$. We need to show only that

CLAIM 2. *For every set $A \subset S_n$ and every $x \in \overline{A}$ (the closure in $\tau|S_n$ one has $f(x) \in \overline{f(A)}$.*

Since A is bounded and $x \in \overline{A}$, then by assumption, there exists a countable subset B of A such that $x \in \overline{B}$. Let F be a τ-closed linear span of B in E. By assumption on f, the restriction $f|F$ is τ-continuous. Hence $f|\overline{B}$ is τ-continuous. Consequently

f is continuous on the $\tau|S_n$-closure of B (since S_n is τ-closed). This implies that

$$f(x) \in \overline{f(B)} \subset \overline{f(A)}.$$

We proved that

$$f|S_n : (S_n, \tau|S_n) \to \mathbb{R}$$

is continuous. But the space E is a (DF)-space, so we conclude that f is τ-continuous and consequently the weak dual of E is realcompact. By Theorem 3 the space $(E, \sigma(E, E'))$ has countable tightness. By Theorem 4 the Mackey space $(E, \mu(E, E'))$ is quasibarrelled. We add a direct proof to this fact:

For every sequence $\alpha := (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $B_\alpha := \bigcap_k n_k S_k^o$. Since E is a (DF)-space, every sequence in any B_α is equicontinuous, so B_α is relatively countably compact in $(E', \sigma(E', E))$. Since $(E', \sigma(E', E))$ is realcompact, one gets that every B_α is relatively compact, hence $\mu(E, E')$-equicontinuous. But every bounded set in $\beta(E', E)$ is contained in some B_α. Hence every $\beta(E', E)$-bounded set is $\mu(E, E')$-equicontinous and this proves that $(E, \mu(E, E'))$ is quasibarrelled.

The last step of the proof is to show that $\tau = \mu(E, E'))$: Indeed, assume on the contrary that τ is strictly weaker than $\mu(E, E')$.

CLAIM 3. *There exists $n \in \mathbb{N}$ such that the topology $\tau|S_n$ is strictly weaker than $\mu(E, E')|S_n$.*

Indeed, if for every $n \in \mathbb{N}$ the equality $\tau|S_n = \mu(E, E')|S_n$ holds, then since E is a (DF)-space one gets that $\tau = \mu(E, E')$, a contradiction.

Hence there exists $n \in \mathbb{N}$ such that our claim holds. Then there exists in S_n a set A which is $\mu(E, E')|S_n$-closed but not $\tau|S_n$-closed. Hence A is not τ-closed. Select

$$x \in \overline{A} \setminus A, \quad x \in S_n$$

(the closure in τ). Since $x \notin A$ (which by assumption is $\mu(E, E')|S_n$-closed), there exists a $\mu(E, E')$-continuous seminorm p on E such that

$$U_x \cap S_n \cap A = \emptyset,$$

where $U_x = \{y \in E : p(x - y) < 1\}$. Hence $p(x - y) \geq 1$ for all $y \in A$. Let (x_n) be any sequence in A. Fix $n \in \mathbb{N}$. There exists a sequence (f_n) of continuous linear functionals on E such that

$$f_n(x - x_n) = 1, \quad |f_n(z)| \leq p(z)$$

for all $z \in E$. This sequence is $\mu(E, E')$-equicontinuous, so $\beta(E', E)$-bounded. Since E is a (DF)-space, the sequence (f_n) is τ-equicontinuous. Therefore the polar $U := \{f_n : n \in \mathbb{N}\}^o$ is a τ-neighbourhood of zero and $(x + 2^{-1}U) \cap \{x_n : n \in \mathbb{N}\} = \emptyset$. But $(x_n) \subset A \subset S_n$, so x does not belong to the τ-closure of the set $\{x_n : n \in \mathbb{N}\}$. This contradicts with the fact that every τ-bounded set has countable tightness. Therefore $\tau = \mu(E, E')$ is quasibarrelled.

Last Theorem applies to get the following interesting

THEOREM 8. *A Fréchet space E is distinguished iff every bounded set in the strong dual has countable tightness.*

This seems to be optimal the best characterization of distinguished spaces in terms of property (2) from Theorem 6.

APPLICATION TO SPACES $C_c(X)$

By $C_c(X)$ we denote the space of all realvalued continuous maps on X endowed with the compact-open topology. Making use of Theorem 4 one gets the following interesting

THEOREM 9 [11]. *The following assertions are equivalent for a dual metric space $C_c(X)$:*

(1) The compact-open topology of $C_c(X)$ is equivalent to the uniform Banach topology generated by the unit ball $[X,1] := \{f \in C_c(X) : f(X) \leqslant 1\}$.

(2) The compact-open topology of $C_c(X)$ has countable tightness.

(3) The weak topology of $C_c(X)$ has countable tightness.

Let ω_1 denote the first ordinal and let $[0,\omega_1)$ be the set of all countable ordinals with the order topology. By $C_c(\omega_1)$ we denote the Morris-Wulber space $C_c([0,\omega_1))$. It is well-known that $C_c(\omega_1)$ is a (DF)-space.

EXAMPLE. *The space $C_c(\omega_1)$ does not have countable tightness for the compact-open topology (and the weak topology). Its weak dual is quasi-Suslin but is not K-analytic.*

$\mathfrak{G}$-BASES AND DUAL METRIC SPACES

Recall that a lcs E admits a $\mathfrak{G}$-basis if there exists a basis of neighbourhoods of zero $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $U_\alpha \subset U_\beta$ for $\beta \leqslant \alpha$ in $\mathbb{N}^\mathbb{N}$. From Theorem 1 it follows that a quasibarrelled space E has a $\mathfrak{G}$-basis iff $E \in \mathfrak{G}$. The weak dual of a lcs having a $\mathfrak{G}$-basis is quasi-Suslin.

The concept of $\mathfrak{G}$-bases is used to present a general argument to construct concrete spaces $C_c(X)$ (different from what Talagrand presented) whose weak dual is not K-analytic but is covered by an ordered family of compact sets. Likely, the first example of this type is due to Valdivia, see [23], who has shown an example of a metrizable and complete lcs E such that $(E'', \sigma(E'', E'))$ is quasi-Suslin and not K-analytic. This applies to show that the strong dual $(E', \beta(E', E))$ of E is a (DF)-space with a $\mathfrak{G}$-basis $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ whose weak dual is quasi-Suslin and not K-analytic, but clearly $(E'', \sigma(E'', E'))$ is covered by an ordered family of compact sets, polars of the sets U_α.

THEOREM 10. *If a lcs E has a closed $\mathfrak{G}$-representation, then $(E', \sigma(E', E))$ is quasi-Suslin. In particular, the weak dual of a dual metric space is quasi-Suslin.*

The concept of $\mathfrak{G}$-basis can nicely be applied to provide a very natural and relatively simple proof (different from what Cascales and Orihuela presented in [5]) for a special case of Cascales-Orihuela result:

THEOREM 11. *In every quasibarrelled lcs in class $\mathfrak{G}$ every precompact set is metrizable.*

Recall that all (LM)-spaces and spaces $D'(\Omega)$, $A(\Omega)$ are quasibarrelled and belong to class $\mathfrak{G}$.

Skech of the proof. Since E is a quasibarrelled space in class $\mathfrak{G}$, then (by Theorem 1) for every $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ there is a bornivorous sequence $(D_{n_1,n_2,\ldots,n_k})_k$ of absolutely convex closed subsets of E such that if $W_\alpha := \bigcup_k D_{n_1,n_2,\ldots,n_k}$, where $\alpha \in \mathbb{N}^\mathbb{N}$, then the family $\{W_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a basis of neighbourhoods of zero in E. Fix a precompact subset P in E.

CLAIM 1. *There exists $n_k \in \mathbb{N}$ and a finite subset F of P such that*

$$P \subset F + 2D_{n_1,n_2,\ldots,n_k}.$$

Indeed, otherwise there exists a sequence $(x_k)_k$ in $\mathbb{N}$ such that

$$x_{k+1} \notin \{x_1, x_2, \ldots x_k\} + 2D_{n_1,n_2,\ldots,n_k}$$

for all $k \in \mathbb{N}$. Since for every $k \in \mathbb{N}$ the last set is closed, there exists a decreasing sequence $(V_k)_k$ of closed absolutely convex neighbourhoods of zero such that $V_{k+1} + V_{k+1} \subset V_k$, $k \in \mathbb{N}$, and that $x_{k+1} \notin \{x_1, x_2, \ldots x_k\} + 2D_{n_1,n_2,\ldots,n_k} + 2V_k$. Set

$$V := \overline{\bigcup_j D_{n_1,n_2,\ldots,n_j} \cap V_{j+1}}.$$

Since E is quasibarrelled and the sequence $(D_{n_1,n_2,\ldots,n_j} \cap V_{j+1})_j$ is bornivorous and $(V_j)_j$ is decreasing, one shows that

$$\overline{\bigcup_j D_{n_1,n_2,\ldots,n_j} \cap V_{j+1}} \subset \bigcup_j 2(D_{n_1,n_2,\ldots,n_j} \cap V_{j+1}) \subset 2D_{n_1,n_2,\ldots,n_k} + 2V_{k+1}$$

for every $k \in \mathbb{N}$. Therefore

$$\operatorname{absconv} V \subset 2D_{n_1,n_2,\ldots,n_k} + 2V_{k+1}$$

for all $k \in \mathbb{N}$. Since the closure W of absconv V is a closed absolutely convex and bornivorous set in E, it is a neighbourhoof of zero (since E is quasibarrelled). But

$$W \subset \overline{2D_{n_1,n_2,\ldots,n_k} + 2V_{k+1}} \subset 2D_{n_1,n_2,\ldots,n_k} + 2V_{k+1} + 2V_{k+1} \subset 2D_{n_1,n_2,\ldots,n_k} + 2V_k.$$

Hence $x_{k+1} \notin \{x_1, x_2, \ldots x_k\} + W$ for all $k \in \mathbb{N}$. This implies that P is not precompact, a contradiction. Claim 1 has been proved.

Now we prove that the precompact set P is metrizable. Choose arbitrary $y \in P$ and let U be an absolutely convex neighbourhood of zero in E. Then there exists a sequence $(n_k)_k$ in $\mathbb{N}$ such that

$$\bigcup_k D_{n_1,n_2,\ldots,n_k} \subset (3/4)U.$$

Using Claim 1 one obtains a finite set F in P and a set $D_{n_1,n_2,\ldots,n_k}$ such that $P \subset F + (1/3)D_{n_1,n_2,\ldots,n_k}$. Fix $x \in F$ such that

$$y \in x + (1/3)D_{n_1,n_2,\ldots,n_k}.$$

The proof will be completed if we show that $x + D_{n_1,n_2,\ldots,n_k}$ intersects P in a relatively neighbourhood of y contained in $y + U$. Define

$$C = \bigcup\{x' + (1/3)D_{n_1,n_2,\ldots,n_k} : x' \in F, y \notin x' + (1/3)D_{n_1,n_2,\ldots,n_k}\}.$$

Observe that

$$P \setminus C \subset y + (2/3)D_{n_1,n_2,\ldots,n_k} \subset x + D_{n_1,n_2,\ldots,n_k}.$$

Clearly $P \setminus C$ is an open neighbourhood of y. On the other hand

$$x + D_{n_1,n_2,\ldots,n_k} = (x + (1/3)D_{n_1,n_2,\ldots,n_k}) + (2/3)D_{n_1,n_2,\ldots,n_k} \subset y + U,$$

and the proof is completed.

The following results describes the cardinality of any $\mathfrak{G}$-basis in nonmetrizable lcs.

THEOREM 12 [11]. *The character $\chi(E)$ of a nonmetrizable lcs E having a $\mathfrak{G}$-basis must satisfy*

$$\mathfrak{b} \leq \chi(E) \leq \mathfrak{d}.$$

Consider the Banach space $\ell^p(\Lambda)$ with the sup-norm topology τ and with closed unit ball D, where p is fixed with $1 \leq p < \infty$ and Λ is an uncountable indexing set. For each $S \subset \Lambda$ define

$$E_S := \{u \in \ell^p(\Lambda) : u(x) = 0, x \notin S\},$$

and for each countable $T \subset \Lambda$ and each $n \in \mathbb{N}$, and define $[n, T] := (n^{-1}D) + E_{\Lambda \setminus T}$. Let E denotes $\ell^p(\Lambda)$ with the locally convex topology ξ having as a base of neighbourhoods of zero all sets of the form $[n, T]$. Note that, for each countable T, the subspaces E_T and $E_{\Lambda \setminus T}$ are topologically complementary in E, and E_T inherits the same Banach topology from E as it does from the Banach space $\ell^p(\Lambda)$, and the dual of E is the same as that of $\ell^p(\Lambda)$.

EXAMPLE. *(E, ξ) is a sequentially complete non-quasibarrelled (DF)-space and does not have countable tightness and whose weak dual is K-analytic.*

Note that $E \in \mathfrak{G}$ for every choice of uncountable Λ, but E admits a $\mathfrak{G}$-basis only when Λ is severely restricted under an axiomatic assumption milder than CH.

EXAMPLE. 1. *If we assume that $\aleph_1 = \mathfrak{d} = |\Lambda|$, then E has a $\mathfrak{G}$-basis.*

2. *If we assume that $\aleph_1 < \mathfrak{b}$, then E does not admit a $\mathfrak{G}$-basis.*

3. *If $|\Lambda| < \mathfrak{b}$ or $|\Lambda| > \mathfrak{d}$, then E does not admit a $\mathfrak{G}$-basis.*

We note the following result of [11].

THEOREM 13 [11]. *If $\aleph_1 < \mathfrak{b}$, then $C_c(\omega_1)$ does not admit a $\mathfrak{G}$-basis.*

Nevertheless for very concrete (DF)-spaces $C_c(\mathfrak{b})$ and $C_c(\mathfrak{d})$ we have the following

THEOREM 14 [11]. *Both spaces $C_c(\mathfrak{b})$ and $C_c(\mathfrak{d})$ have a $\mathfrak{G}$-basis. If the cardinal κ has cofinality $\aleph_0$, $\mathfrak{b}$, or $\mathfrak{d}$, then $C_c(\kappa)$ has a $\mathfrak{G}$-basis.*

Combining previous results, we obtain

THEOREM 15 [11]. *The Morris-Wulbert space $C_c(\omega_1)$ has a $\mathfrak{G}$-basis iff $\aleph_1 = \mathfrak{b}$.*

SPACES $C_p(X)$ OVER METRIC SPACES X

Very recently Cascales and Namioka [9] proved that for a K-analytic space X the following conditions are equivalent:

(1) $C_p(X)$ is Fréchet-Urysohn.

(2) $C_p(X)$ is a k_R-space.

(3) X is σ-scattered.

(4) Every countable subset of $C_p(X)$ has metrizable closure.

In 1981 McCoy asked [16] if every first countable space X for which $C_p(X)$ is Fréchet-Urysohn must be countable. This motivated the following our recent result:

THEOREM 16 [12]. *For a metric and complete space X the following assertions are equivalent:*

(1) $C_p(X)$ is Fréchet-Urysohn.

(2) $C_p(X)$ has bounded tightness.

(3) X is separable and every compact subset of X is scattered.

(4) X is separable and scattered.

(5) X is countable, i.e. the space $C_p(X)$ is metrizable.

(6) X is separable and every countable subset of $C_p(X)$ has metrizable closure.

On the other hand, for spaces $C_c(X)$ we note the following

THEOREM 16 [12]. *If X is a locally compact unbounded metric space, then the following assertions are equivalent:*

(1) $C_c(X)$ is a Fréchet space.

(2) $C_c(X)$ has bounded tightness.

(3) $C_c(X)$ has countable tightness.

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