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ON THE FIXED-POINT SET OF AN AUTOMORPHISM OF A CLOSED NONORIENTABLE SURFACE

On the fixed-point set of an automorphism of a closed nonorientable surface

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Abstract

Macbeath gave a formula for the number of fixed points for each non-identity automorphism of a compact Riemann surface in terms of the universal covering transformation group of the cyclic group generated by the automorphism. This formula generalizes to determine the fixed point set of each non-identity automorphism acting on a closed non-orientable surface with one exception; namely, when this element has order 2. In this case the fixed point set may have simple closed curves (called *ovals*) as well as fixed points. Izquierdo and Singerman explained how Macbeath's results generalize to automorphisms of a nonorientable surface with orden distinct from 2 and also determined the fixed point set of an automorphism of order 2. In this paper we will discuss certain extensions of these results.

1 Preliminaries on NEC groups and their quotient orbifolds

A *non-euclidean crystallographic group* (NEC) group is a discrete group Γ of the group $\mathcal{G}$ of isometries of the hyperbolic plane $\mathcal{H}$ with compact quotient space $\mathcal{H}/\Gamma$. If the group Γ is a subgroup of the group $\mathcal{G}^+$ of orientation-preserving isometries of $\mathcal{H}$, then it is called a *Fuchsian group.* Otherwise $\Gamma^+ = \Gamma \bigcap \mathcal{G}^+$ is a subgroup of index 2 in Γ called its *canonical Fuchsian.*

An NEC group is determined by its signature

$$s(\Gamma) = (h; \pm; [m_1, ..., m_r]; \{(n_{11}, ..., n_{1s_1}), ..., (n_{k1}, ..., n_{ks_k})\}). \tag{1}$$

The quotient space $\mathcal{H}/\Gamma$ is an orbifold with underlying surface of genus h with r cone points, each of order m_i, and k mirror lines, each with $s_i \geq 0$ corner points each of order n_{ij}. The signs $+$ or $-$ correspond to orientable or non-orientable orbifolds respectively. Γ is called the *group* (or fundamental group) of the orbifold $\mathcal{H}/\Gamma$

Associated to the signature (1) there is a presentation for Γ with generators

$$x_1, ..., x_r, e_1, ..., e_k, c_{ij}, 1 \leq i \leq k, 1 \leq j \leq s_i,$$

$a_1, b_1, ..., a_h, b_h$ if $\mathcal{H}/\Gamma$ is orientable or $a_1, ..., a_h$, if $\mathcal{H}/\Gamma$ is non-orientable.

and relators

$$x_i^{m_i}, i = 1, ..., r, c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, i = 1, ..., k, j = 1, ..., s_i, c_{i0}e_i^{-1}c_{is_i}e_i,$$

$$x_1x_2...x_re_1...e_ka_1b_1a_1^{-1}b_1^{-1}...a_h^{-1}b_h^{-1}, \text{ if } \mathcal{H}/\Gamma \text{ is orientable or}$$
$$x_1x_2...x_re_1...e_ka_1^2...a_h^2, \text{ if } \mathcal{H}/\Gamma \text{ is non-orientable.} \quad (2)$$

In these presentations, the only elements of finite order are the elliptic elements and the reflections. The elliptic elements are conjugate of powers of the x_i or $c_{ij-1}c_{ij}$ and the reflections are conjugate of the c_{ij}. The e_i generators are orientation preserving. They are called the *connecting generators.* Each period cycle corresponds to a conjugacy class of reflections in Γ.

Let Γ be an NEC group with presentation (2). Then there is a fundamental region P for Γ which is a polygon in $\mathcal{H}$ whose perimeter, described counter-clockwise, is one of the following according to the orientability of $\mathcal{H}/\Gamma$:

$$\epsilon_1\epsilon_1' \dots \epsilon_r\epsilon_r' \dots \delta_1\gamma_{10} \dots \gamma_{1s_1}\delta_1' \dots \delta_k\gamma_{k0} \dots \gamma_{ks_k}\delta_k'\alpha_1\beta_1\alpha_1'\beta_1' \dots \alpha_h\beta_h\alpha_h'\beta_h' \quad (3)$$

$$\epsilon_1\epsilon_1' \dots \epsilon_r\epsilon_r' \dots \delta_1\gamma_{10} \dots \gamma_{1s_1}\delta_1' \dots \delta_k\gamma_{k0} \dots \gamma_{ks_k}\delta_k'\alpha_1\alpha_1^* \dots \alpha_h\alpha_h^* \quad (4)$$

The elliptic element x_i in Γ pairs the sides ϵ_i and ϵ_i' of P, the generating reflection c_{ij} has axis containing δ_{ij}, the element e_i pairs the sides ϵ_i and ϵ_i', and the elements a_j and b_j pair the sides α_j and α_j' (α_j^* in the non-orientable case) and β_j and β_j'. The quotient space $P/ <$ pairings $>$ is an orbifold which is isomorphic to the orbifold $\mathcal{H}/\Gamma$.

The hyperbolic area of the orbifold $\mathcal{H}/\Gamma$ is:

$$\mu(P) = \mu(\Gamma) = 2\pi(\epsilon h - 2 + k + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - \frac{1}{n_{ij}})), \quad (5)$$

where $\epsilon = 2$ if there is a $+$ sign and $\epsilon = 1$ if there is a $-$ sign. If Γ^* is a subgroup of Γ of finite index then the Riemann-Hurwitz formula holds:

$$|\Gamma : \Gamma^*| = \frac{\mu(\Gamma^*)}{\mu(\Gamma)}. \quad (6)$$

Let Γ^+ be the canonical Fuchsian of an NEC group Γ. Then the orbifold H/Γ^+ is a 2-sheeted covering of $\mathcal{H}/\Gamma$, called its *complex double.* The genus $h^+ = \epsilon h + k - 1$ of $\mathcal{H}/\Gamma^+$ is the *algebraic genus* of $\mathcal{H}/\Gamma$. An NEC group K without elliptic elements is called a *surface group.* Its signature is $(g; \pm; [\ \ ]; \{(\ \ )^k\})$. A Klein surface whose complex double has genus greater than one can be expressed as $\mathcal{H}/K$ where K is an NEC surface group. An orientable Klein surface without boundary can be thought as a *Riemann surface.*

If G is a finite group, then G is a group of automorphisms of a Klein surface $Y = \mathcal{H}/K$ if and only if there exists an NEC group Γ and a smooth homomorphism

$$\theta : \Gamma \to G \tag{7}$$

having K as the kernel. The transformation group $(\Gamma, \mathcal{H})$ is called the *universal covering transformation group* of (G, Y).

2 The universal covering transformation group

From now on, given a compact non-orientable Klein surface Y of genus $p \geq 3$, the universal covering space of Y is the hyperbolic plane $\mathcal{H}$ and the group of covering transformations is a surface subgroup K generated by glide-reflections. Given a group G of automorphisms of Y, let Γ denote its lifting to $\mathcal{H}$ and let $\theta : \Gamma \to G$ be the monodromy, as given in (7), for the covering $p : Y \to Y/G = \mathcal{H}/\Gamma$.

We shall consider cyclic groups G of automorphisms of Y and find the conditions that they impose over the universal covering transformation groups Γ.

We begin by considering a cyclic group $G =< t | t^N = 1 >$ of automorphisms of Y of odd order N. Γ is the fundamental group of the quotient orbifold Y/G, then, as θ is smooth, i.e. it preserves the orders of the elements in Γ, we must have $o(\theta(g)) \equiv 1(2)$ for every element g of finite order in Γ. Also we cannot have period cycles in $s(\Gamma)$. Thus Γ has signature (1) of the form

$$s(\Gamma) = (g; -; [m_1, ..., m_n]; \{\ \ \}) \tag{8}$$

with $m_i, 1 \leq i \leq n$ a divisor of N.

Then Γ has presentation

$$< x_1, \ldots, x_n, d_1, ..., d_g \mid x_i^{m_i} = 1, i = 1, ..., n, x_1 ... x_n d_1^2 .. d_g^2 > \tag{9}$$

It means that the fixed-point set of the automorphism t consists of a finite number of points in Y. (see [3], [6])

Theorem 2.1 *Let Y be a non-orientable surface of topölogical genus $p \geq 3$. Let $G \cong C_N =< t \mid t^N = 1 >$, with N odd, be a group of automorphisms of Y. The fixed-point set of an automorphism t^i of order d in G consists of a finite number of fixed points. This number is given by Macbeath's formula (see [6])*

$$N \sum_{d|m_j} \frac{1}{m_j} \tag{10}$$

Proof The universal covering transformation group Γ associated to (G, Y) is given in 8 and 9. Now, we are in the hypothesis of Macbeath's result [6].

This is because Macbeath's proof (applying to orientable quotient spaces) only uses the facts that each proper period corresponds to a unique conjugacy class of elliptic elements of Γ, and each elliptic element has a unique fixed point in $\mathcal{H}$.

Now, let $G =< t | t^{2N} = 1 >$ be a cyclic group of automorphisms of Y of order $2N$, and let Γ be the fundamental group of the quotient orbifold Y/G. As θ is smooth, we must have $\theta(c) = t^N$ for every reflection c in Γ. Also we cannot have two distinct reflections in Γ whose product has finite order. So it follows that, in the canonical signature of NEC groups as given in (1), Γ has empty period cycles.

Thus Γ has signature of the form

$$s(\Gamma) = (g; \pm; [m_1, ..., m_n]; \{(\ \ )^k\}) \tag{11}$$

with k empty period cycles. Then Γ has one of the two presentations depending on whether there is a $+$ or a $-$ in the signature;

for the $(+)$ case

$$\begin{gathered} x_1, \ldots, x_n, e_1, \ldots, e_k, c_1, \ldots, c_k, a_1, b_1, \ldots, a_g, b_g \mid \\ x_i^{m_i} = 1, i = 1, ..., n, c_j^2 = c_j e_j^{-1} c_j e_j = 1, j = 1, ..., k, \\ x_1 ... x_n e_1 ... e_k a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a g h^{-1} b_g^{-1} \end{gathered} \tag{12}$$

for the $(-)$ case

$$\begin{gathered} x_1, \ldots, x_n, e_1, \ldots, e_k, c_1, \ldots, c_k, d_1, ..., d_g \mid \\ x_i^{m_i} = 1, i = 1, ..., n, c_j^2 = c_j e_j^{-1} c_j e_j = 1, j = 1, ..., k, x_1 ... x_n e_1 ... e_k d_1^2 ... d_g^2 \end{gathered} \tag{13}$$

One important fact to note about these presentations is that the connecting generator e_j commutes with the generating reflection c_j, and in fact the centralizer of c_j in Γ is just the group $gp < c_j,\ e_j >= C_2 \times C_\infty$. (See [9])

3 The fixed-point set of a power of an automorphim t of even order

Let Y be a non-orientable surface of topological genus $p \geq 3$ and let t be an automorphism of order $2N$. If $1 \leq i < 2N$ and $i \neq N$ then the fixed-point set of the automorphism t^i consists of a finite number of fixed points, since the possible generating reflections in the subgroup $\Lambda = \theta^{-1}(< t^i >)$ are mapped by θ to t^N (see [3]). Again, the number of fixed points of the automorphism t^i is given by Macbeath's formula (see [6]). This number is given in the following

Theorem 3.1 *Let Y be a non-orientable surface of topological genus $p \geq 3$. Let $G \cong C_N =< t \mid t^{2N} = 1 >$ be a group of automorphisms of Y. The fixed-point set of an automorphism t^i, $i \neq N$ of order d in G consists of a finite number of fixed points. This number is given by Macbeath's formula (see [6])*

$$2N \sum_{d|m_j} \frac{1}{m_j} \tag{14}$$

fixed points, where m_j runs over the periods in $s(\Gamma)$, Γ being the universal covering transformation group of (G, Y).

Notice that the number of isolated fixed points of t^i is independent of the smooth epimorphism θ above. However the epimorphism θ does play a part in the number of ovals of t^N.

Theorem 3.2 [4] *Let Y be a non-orientable surface of topological genus $p \geq 3$. Let $G \cong C_{2N} =< t \mid t^{2N} = 1 >$ be a group of automorphisms of Y, and let θ and Γ be as described in equations 7 and 11. If $\theta(e_j) = t^{v_j}$ than the number of ovals of the involution t^N is*

$$\sum_{j=1}^{k} (N, v_j) \tag{15}$$

and the number of isolated fixed points of t^N is

$$2N \sum_{m_j \ even} \frac{1}{m_j}$$

Proof. Let $\Lambda = \theta^{-1}(< t^N >)$ so that Λ contains the group $K = Ker\theta$ with index 2. Now, Λ must have signature of the form

$$s(\Lambda) = (g; \pm; [2^{(r)}]; \{(\ \ )^s\}) \tag{16}$$

with r periods equal to 2 and s empty period cycles.

The group monomorphisms

$$\{1_d\} \longrightarrow C_2 \longrightarrow C_{2N} \tag{17}$$

$$K \longrightarrow \Lambda \longrightarrow \Gamma \tag{18}$$

yield us the following (orbifold-)coverings

$$Y \longrightarrow Y/G_1 = \mathcal{H}/\Lambda \longrightarrow Y/G = \mathcal{H}/\Gamma \tag{19}$$

The group $G_1 =< t^N >$ is a group of automorphisms of Y with 2 automorphisms.

By results in [2] (see also [3]), r is the number of isolated fixed points of t^N and is given by Macbeath's formula

$$2N \sum_{m_j \ even} \frac{1}{m_j}$$

It also follows from [2] that the number of ovals of t^N is just the number s of period cycles in Λ, which corresponds to the number of conjugacy classes of reflections in Λ. As a reflection c_j in Λ belongs also to Γ and the group Γ has k conjugacy classes of reflections, we just have to determine into how many Λ-conjugacy classes the Γ-conjugacy class of c_j splits. We shall use the epimorphism θ to calculate this number.

There is a transitive action of Γ on the Λ-conjugacy classes of c_j in Λ by letting $\gamma \in \Gamma$ map the reflection gc_jg^{-1} to $g\gamma c_j\gamma^{-1}g^{-1}$, with $g \in \Lambda$. (Because $\Lambda \lhd \Gamma$). Clearly, if $\lambda \in \Lambda$ then λ has a trivial action on these Λ-conjugacy classes. So we have an action of $\Gamma/\Lambda \cong C_{2N}/C_2 \cong C_N$ on these classes. As the centralizer of c_j in Γ is just $< c_j, e_j >$, the stabilizer of the Λ-conjugacy classes of c_j in Λ are the cosets $\Lambda, \Lambda e_j, \ldots, \Lambda e_j^{\delta_j - 1}$, where $\delta_j = exp_\Lambda e_j$, the least positive power of e_j that belongs to Λ. Now, let $\varepsilon_j = exp_K e_j$. Then either $\varepsilon_j = \delta_j$ or $\varepsilon_j = 2\delta_j$.

The additive group Z_{2N} contains a subgroup isomorphic to Z_N and $a \in Z_N$ has order $\frac{N}{(N,a)}$ in Z_N so that a has the same order in Z_{2N} if and only if $(2N, a) = 2(N, a)$. If $(2N, a) = (N, a)$ then the order of a in Z_{2N} is twice the order of a in Z_N and we then find that

$$\varepsilon_j = \delta_j \qquad \text{if} \qquad (2N, v_j) = 2(N, v_j)$$

and

$$\varepsilon_j = 2\delta_j \qquad \text{if} \qquad (2N, v_j) = (N, v_j),$$

where $\theta(e_j) = t^{v_j}$.

By the above argument on the action of Γ/Λ on the Λ-conjugacy classes of c_j we see that the number of such classes is N/δ_j, which is

if $\varepsilon_j = \delta_j$

$$\frac{N}{\delta_j} = \frac{N}{\varepsilon_j} = \frac{N(2N, v_j)}{2N} = \frac{(2N, v_j)}{2} = (N, v_j),$$

or if $\varepsilon_j = 2\delta_j$

$$\frac{N}{\delta_j} = \frac{2N}{\varepsilon_j} = \frac{2N(2N, v_j)}{2N} = (2N, v_j) = (N, v_j)$$

Thus in both cases the generating reflection c_j of Γ induces (N, v_j) conjugacy classes of reflections in Λ. Thus the number of ovals of t^N in Y is

$$\sum_{j=1}^{k} (N, v_j) \tag{20}$$

Theorem 3.3 [4] *The ovals of t^N in Y induced by the jth period cycle in Γ are twisted if $(2N, v_j) = (N, v_j)$ and untwisted if $(2N, v_j) = 2(N, v_j)$.*

Proof. As we have found in Theorem 3.2, The jth empty period cycle in Γ induces (N, v_j) empty period cycles in Λ. The generating reflections of these period cycles are just conjugates of c_j in Γ and, as the corresponding connecting generator e_j is just the orientation-preserving element generating the centralizer of c_j in Γ, we see that the connecting generator of each of the period cycles in Λ induced by the jth period cycle in Γ is just conjugate to $e_j^{\delta_j}$, $\delta_j = exp_\Lambda e_j$ as in the proof of Theorem 3.2. Now, let $\theta' : \Lambda \to C_2 = gp < \xi >$, where $\xi = t^N$, be the restriction of the epimorphism $\theta : \Gamma \to C_{2N}$. Then

if $\varepsilon_j = \delta_j$

$$\theta'(e_j^{\delta_j}) \quad = \quad \theta'(e_j^{\varepsilon_j}) \quad = \quad \theta(e_j^{\varepsilon_j}) \quad = \quad 1$$

if $\varepsilon_j = 2\delta_j$

$$\theta'(e_j^{\delta_j}) \quad = \quad \theta'(e_j^{\frac{\varepsilon_j}{2}}) \quad = \quad \theta(e_j^{\frac{\varepsilon_j}{2}}) \quad = \quad \xi,$$

ξ the generator of C_2. Generally, if c is the generating reflection of an empty period cycle of Λ and e is the corresponding connecting generator then figures 1 and 2 show that $\theta'(e) = 1$ corresponds to an untwisted oval while $\theta'(e) = \xi$ corresponds to a twisted oval.

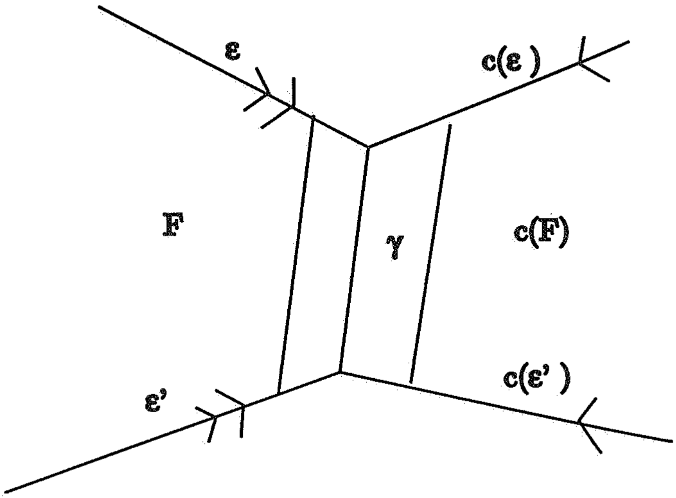

Figure 1: $\theta'(e) = 1$ so $e \in K$

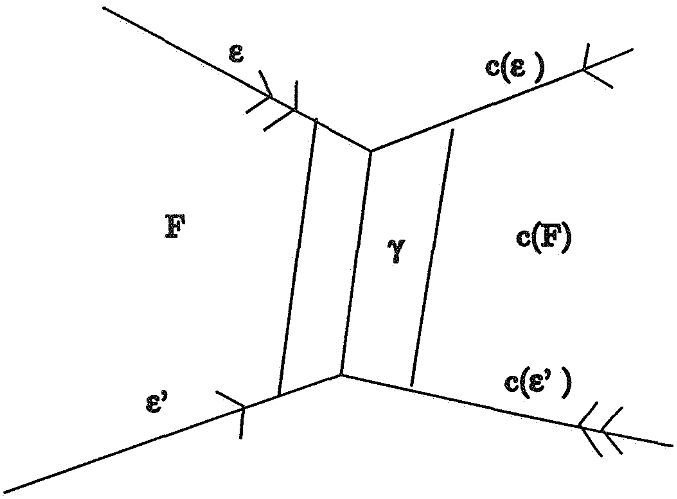

Figure 2: $\theta'(e) = \xi$ so $ce \in K$

However, as in the proof of Theorem 3.2 $\varepsilon_j = \delta_j$ if and only if $(2N, v_j) = 2(N, v_j)$ and hence we have untwisted ovals while $\varepsilon_j = 2\delta_j$ if and only if $(2N, v_j) = (N, v_j)$ and we have twisted ovals.

4 Bounds and examples

In [7] (also see [2]) Scherrer showed that that if an involution of a non-orientable surface of genus p has $\mid F \mid$ fixed points and $\mid V \mid$ ovals then

$$\mid F \mid + 2 \mid V \mid \leq p + 2.$$

In our examples we will show that for any integer N we can find a non-orientable surface of genus p admitting a C_{2N} action with generator t such that t^N attains the Scherrer bound.

Example 1. [4] Bujalance [1] found the maximum order for an automorphism t of a non-orientable surface Y of genus $p \geq 3$; it is $2p$ for odd p and $2(p-1)$ for even p. The universal covering transformation group Γ has signature $s(\Gamma) = (0; [2, p]; \{(\ \ )\})$ for odd p, and signature $s(\Gamma) = (0; [2, 2(p-1)]; \{(\ \ )\})$ for even p. There is, essentially, only one way of defining the epimorphism θ in each case:

if p is odd, we define $\theta : \Gamma \to C_{2p}$ by $\theta(x_1) = t^p$, $\theta(x_2) = t^2$, $\theta(c) = t^p$, and $\theta(e) = t^{p-2}$,

if p is even, we define $\theta : \Gamma \to C_{2(p-1)}$ by $\theta(x_1) = t^{p-1}$, $\theta(x_2) = t^1$, $\theta(c) = t^{p-1}$, and $\theta(e) = t^{p-2}$.

Using Macbeath's formula (14) we see that the involution t^p has p fixed points for surfaces of both odd and even genera. Now, if p is odd then the involution t^p also has, by Theorems 3.2 and 3.3, one twisted oval if p is odd as $(p, p-2) = (2p, p-2) = 1$. If p is even then the involution t^{p-1} has, by Theorems 3.2 and 3.3, one untwisted oval as $(p-1, p-2) = 1$ and $(2(p-1), p-2) = 2(p, p-2) = 2$. We note that the involution t^p obeys the Scherrer bound. Note that the orders of the cyclic groups in Bujulance's examples are $\equiv 2 \mod 4$. Our second example shows that the Scherrer bound can be obtained for the involution in a C_{4m} action.

Example 2. Let Y be a non-orientable surface of genus $p = 4mk \geq 3$, and let t be an automorphism of Y of order $4m$. Let Γ have signature

$$(0; +; [4m, 4m]; (\quad)^k)$$

and define a smooth epimorphism $\theta : \Gamma \to C_4$ by mapping the two generators of order $4m$ to t and t^{-1} and the connecting generators to the identity. By Theorems 3.2 and 3.3 all the ovals of t^{2m} are untwisted. We then find that for the involution t^{2m}, $\mid F \mid= 2$, and $\mid V \mid= 2mk$,and $p + 2 = 4mk + 2$, so that we find infinitely many surfaces where the Scherrer bound is attained for the involution in C_{4m}.

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