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FUCHSIAN GROUPS AND UNIFORMIZATION OF HURWITZ SPACES

Fuchsian groups and uniformization of Hurwitz spaces

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Recall that a meromorphic function is a pair (P, f), where P is a Riemann surface and $f : P \to \bar{\mathbb{C}}$ is a holomorphic map to the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$. We say that meromorphic functions (P_1, f_1) and (P_2, f_2) are *biholomorphic equivalent* if there exist biholomorphic maps $\varphi_P : P_1 \to P_2$ and $\varphi_{\bar{\mathbb{C}}} : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $\varphi_{\bar{\mathbb{C}}} f_1 = f_2 \varphi_P$.

We say that (P_1, f_1) and (P_2, f_2) have *the same topological type* if there exist homeomorphisms $\varphi_P : P_1 \to P_2$ and $\varphi_{\bar{\mathbb{C}}} : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $\varphi_{\bar{\mathbb{C}}} f_1 = f_2 \varphi_P$.

In this paper we prove that for each (P, f) the space of biholomorphic equivalent classes of meromorphic functions with the same topological type as (P, f) has a representation $\mathbb{R}^m/\text{Mod}$, where Mod is a discrete group.

For functions of general positions this was proved in [2]. For trigonometric polynomial this was proved in [1].

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1. Geometry of Fuchsian groups

Let

$$\Lambda = \{z \in \mathbb{C} \mid \text{Im} \quad z > 0\}$$

be the Lobachevsky's plane. The group of its automorphisms $\text{Aut}(\Lambda)$ consists of maps

$$z \mapsto \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. An automorphism is called *elliptic* if it has one fixed point in Λ. An automorphism is called *parabolic* if it has one fixed point in R. An automorphism is called *hyperbolic* if it has two fixed points in R. The set $\text{Aut}(\Lambda)\backslash 1$ are divided on three subset:

a set $\text{Aut}_+(\Lambda)$ of elliptic automorphisms,

a set $\text{Aut}_0(\Lambda)$ of parabolic automorphisms,

a set $\text{Aut}_-(\Lambda)$ of hyperbolic automorphisms.

Each $C \in \text{Aut}_0(\Lambda)$ is of the forme

$$C(z) = \frac{(1 - a\gamma)z + a^2\gamma}{-\gamma z + (1 + a\gamma)},$$

where $C(a) = a$.

It is called *positive* if

$$\frac{1 - a\gamma}{-\gamma} < a$$

and *negative* if

$$\frac{1 - a\gamma}{-\gamma} > a.$$

Each $C \in \mathrm{Aut}_-(\Lambda)$ is of the forme

$$C(z) = \frac{(\lambda\alpha - \beta)z + (1-\lambda)\alpha\beta}{(\lambda - 1)z + (\alpha - \lambda\beta)},$$

where $\lambda > 1$, $C(\alpha) = \alpha$, $C(\beta) = \beta$. It is called *positive* if $\alpha < \beta$ and *negative* if $\alpha > \beta$.

Let

$$C_1, C_2 \in \mathrm{Aut}_0(\Lambda) \cup \mathrm{Aut}_-(\Lambda).$$

We say that $C_1 < C_2$ if all fixed points of C_1 are less that all fixed points of C_2.

The set

$$\{C_1, C_2, C_3\} \in \mathrm{Aut}_0(\Lambda) \cup \mathrm{Aut}_-(\Lambda)$$

is called *sequential* if

$$C_1 C_2 C_3 = 1$$

and there exists

$$D \in \mathrm{Aut}(\Lambda)$$

such that

$$\widetilde{C}_i = D C_i D^{-1} \quad (i = 1, 2, 3)$$

are positive and

$$\widetilde{C}_1 < \widetilde{C}_2 < \widetilde{C}_3.$$

Lemma 1. *Let*

$$C_1 z = \lambda z \quad (\lambda > 1),$$

$$C_2(z) = \frac{(1 - a\gamma)z + a^2\gamma}{-\gamma z + (1 + a\gamma)}$$

and

$$C_3 = (C_1 C_2)^{-1}.$$

Then $\{C_1, C_2, C_3\}$ *is sequential if and only if* $a > 0$ *and*

$$a\gamma \geq \frac{\sqrt{\lambda} + 1}{\sqrt{\lambda} - 1}.$$

Further

$$C_3 \in Aut_0(\Lambda)$$

if and only if

$$a\gamma = \frac{\sqrt{\lambda} + 1}{\sqrt{\lambda} - 1}.$$

Proof. We have

$$C_3^{-1} z = C_1 C_2 z = \frac{(1 - a\gamma)\lambda z + a^2\gamma\lambda}{-\gamma z + (1 + a\gamma)}.$$

Fixed points of C_3 are solutions of the equation

$$\gamma x^2 + (\lambda - a\gamma\lambda - 1 - a\gamma)x + a^2\gamma\lambda = 0. \tag{1}$$

Thus the condition

$$C_3 \in \mathrm{Aut}_0(\Lambda) \cup \mathrm{Aut}_-(\Lambda)$$

is equivalent to

$$a\gamma \notin (\frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1}, \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}).$$

Further

$$C_3 \in \mathrm{Aut}_0(\Lambda)$$

if and only if

$$a\gamma = \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1}$$

or

$$a\gamma = \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}.$$

Suppose that

$$\{C_1, C_2, C_3\}$$

is a sequential. Then $a > 0$, $\gamma > 0$ and the roots of (1) are more that a. Thus

$$\frac{a\gamma\lambda + a\gamma + 1 - \lambda}{\gamma} > a$$

and

$$a\gamma > \frac{\lambda - 1}{\lambda} > \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1}.$$

Therefore

$$a\gamma \geq \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}.$$

Conversely. Suppose that

$$a > 0, \quad a\gamma \geq \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1};$$

then

$$\gamma > 0, \qquad \frac{1 - a\gamma}{-\gamma} < a$$

and therefore

$$C_3 > C_2.$$

Finally

$$C_3^{-1}(\infty) = \frac{1 - a\gamma}{-\gamma} > 0$$

and thus C_3 is positive. □

The set

$$\{C_1, ..., C_n\}$$

is called *sequential* if the set

$$\{C_1 \cdots C_{i-1}, C_i, C_{i+1} \cdots C_n\}$$

is sequential for $i = 2, \ldots, n-1$.

The following statement is a consequence of [3].

Lemma 2. *1) Any sequential set*

$$\{C_1, ..., C_n\}$$

generates a free Fuchsian group

$$\Gamma \subset Aut_0(\Lambda) \cup Aut_-(\Lambda) \cup 1$$

such that Λ/Γ *is a sphere with* k_0 *punctures and* k_- *holes, where* k_0 *(respectively* k_-*) is the number of parabolic (respectively hyperbolic) transformations among* C_i.

2) Any sphere with k_0 *punctures and* k_- *holes is biholomorphic equivalent to* Λ/Γ*, where* Γ *is a Fuchsian group genereted by the sequential set* $\{C_1, ..., C_n\}$.

2. Moduli spaces of meromorphic functions

Denote by (S_k, δ_k) any pair, where S_k is the free group with free generators

$$c_1, \ldots, c_{k-1}$$

and

$$\delta_k = \{c_1, \ldots, c_{k-1}, c_k\},$$

where

$$c_k = (c_1 \cdots c_{k-1})^{-1}.$$

Any monomorphism

$$\psi : S_k \to \mathrm{Aut}(\Lambda)$$

is called a *realization* of (S_k, δ_k) if

$$\{\psi(c_1), \ldots, \psi(c_k)\}$$

is a sequential set of parabolic elements.

Let T_k^* be a set of all realizations (S_k, δ_k). The group $\mathrm{Aut}(\Lambda)$ acts in T_k^* by conjugation

$$\psi(c_i) \mapsto \gamma\psi(c_i)\gamma^{-1} \quad (\gamma \in \mathrm{Aut}(\Lambda)).$$

Put

$$T_k = T_k^*/\mathrm{Aut}(\Lambda).$$

There exists a natural bijection between T_k and

$$T'_k = \{\psi \in T_k^* \mid \psi(c_1c_2)^{-1}z = \lambda z, \quad \lambda > 1, \quad \psi(c_2)(-1) = -1\}.$$

We note that $T_k \cong T'_k$ is embedded in $\mathbb{R}^{2k-6}$ by a map

$$\psi \mapsto \{\lambda, \gamma_3, a_3, \gamma_4, a_4, \ldots, a_{k-2}, \gamma_{k-1}\} \in \mathbb{R}^{2k-6},$$

where

$$\psi(c_i)z = \frac{(1 - a_i\gamma_i)z + a_i^2\gamma_i}{-\gamma_i z + (1 + a_i\gamma_i)}.$$

Theorem 1.

$$T_k \cong \mathbb{R}^{2k-6}.$$

Proof: Lemma 1 gives a discription of

$$\psi \in T'_k.$$

These conditions have the forme $\lambda > 1, \quad a_i > b_i$ and $a_i\gamma_i > v_i$, where

$$b_i = b_i(\lambda, \gamma_3, a_3, \ldots, \gamma_{i-1}, a_{i-1}),$$

$$v_i = v_i(\lambda, \gamma_3, a_3, \ldots, \gamma_{i-1}, a_{i-1}).$$

This proves the theorem. □

Let M_k be the moduli space (i.c., that the space of classes of biholomorphic equivalents) of spheres with $k > 3$ punctures.

Theorem 2 [4].

$$M_k = T_k/\mathrm{Mod}_k \cong \mathbb{R}^{2k-6}/\mathrm{Mod}_k,$$

where Mod_k acts discretly on T_k.

Proof: Let

$$\Psi^* : T_k^* \to M_k$$

be the map such that

$$\Psi^*(\psi) = \Lambda/\psi(S_k).$$

It gives

$$\Psi : T_k \to M_k.$$

From Lemma 2 it follows that Ψ is surjective.

Moreover,

$$\Psi(\psi_1) = \Psi(\psi_2)$$

if and only if there exists a $\gamma \in \mathrm{Aut}(\Lambda)$, a permutation

$$\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$$

and

$$\alpha_i \in S_k \quad (i = 1, \dots, k)$$

such that

$$\psi_1(S_k) = \gamma\psi_2(S_k)\gamma^{-1}$$

and

$$\psi_1(c_i) = \gamma\psi_2(\alpha_i c_{\sigma(i)} \alpha_i^{-1})\gamma^{-1}.$$

The sets $(\gamma, \sigma, \alpha_i)$ form a group B_k^*. It acts on T_k and $M_k = T_k/B_k^*$. The sets $(\gamma, 1, 1, \dots)$ form a normal subgroup

$$B_k^0 \subset B_k^*,$$

which is the kernel of the action of B_k^*. Thus

$$B_k = B_k^*/B_k^0$$

acts on T_k and

$$M_k = T_k/B_k.$$

Let $\psi \in T_k$. The metric on Λ determines a metric on $P = \Lambda/\psi(S_k)$ with constant negative curvature. Each $c \in S_k$ gives a unique geodesic curve $\widetilde{c} \subset P$. For $P \subset M_k$ of general position, lengths $\rho(c)$ are different for different c. The system of generators allows us to enumerate all elements $c \in S_k$ and thus we have a correspondence

$$P \mapsto v = (\rho(c_1), \rho(c_2), \dots) \in \mathbb{R}^\infty.$$

The coordinates v form a discrete set in $\mathbb{R}$ and thus B_k acts discrete with respect to any sensible topology in $\mathbb{R}^\infty$. Moreover $\rho(c_i)$ is algebraically expressed in the coordinats

$$(\lambda, \gamma_3, a_3, \dots, \gamma_{k-1})$$

on T_k. Thus B_k acts discretely on T_k in our topology. According to Theorem 1, $T_k \cong \mathbb{R}^{2k-6}$.

3. Uniformisation of meromorphic functions

Let (P, f) be a meromorphic function. The genus g of P is called the *genus* of (P, f) and the number of sheets of f is called *degree* of (P, f). A point $p \in P$ is called a *singular point* if $df(p) = 0$. The image $f(p) \in \bar{\mathbb{C}}$ of a singular point is called a *singular value.*

Let

$$a_1, \dots, a_k \in \bar{\mathbb{C}}$$

be singular values of f. Let

$$b_i^1, \dots, b_i^{m_i}$$

be singular points, corresponding to a_i. Let n_i^s be the number of branchs in b_i^s. Then

$$\sum_{s=1}^{m_i} n_i^s \leq n.$$

From Riemann-Hurwitz formula it follows that

$$2(g-1) = -2n + \sum_{i,s}(n_i^s - 1) \leq -2n + (kn - \sum_{i=1}^{k} m_i) \leq -2n + k(n-1)$$

and thus

$$2(g+n-1) \leq k(n-1),$$

$$k \geq 2\frac{n-1+g}{n-1} = 2 + 2\frac{g}{n-1}.$$

If $k = 2$ then $g = 0$, $m_1 = m_2 = 1$ and thus (P, f) is equivalent $(\bar{\mathbb{C}}, z^n)$, where $z^n : z \mapsto z^n$.

Further we assume that $k \geq 3$.

Let $S^n \subset S_k$be a subgroup of index n and $\psi \in T_k^*$. Put

$$\widetilde{P} = \Lambda/\psi(S^n), \quad \widetilde{\mathbb{C}} = \Lambda/\psi(S_k).$$

The inclusion $S^n \subset S_k$ gives a holomorphic covering $\widetilde{f} : \widetilde{P} \to \widetilde{\mathbb{C}}$. Let P and $\bar{\mathbb{C}}$ be natural compactifications of the punctured surfaces $\widetilde{P}$ and $\widetilde{\mathbb{C}}$. Then $\widetilde{f}$ has a continuation $f : P \to \bar{\mathbb{C}}$ and (P, f) is a meromorphic function. Put

$$\Psi_{S^n}(\psi) = (P, f).$$

Lemma 3. *Let (P, f) be a meromorphic function. Then there exists a subgroup $S^n \subset S_k$ and $\psi \in T_k$ such that*

$$\Psi_{S^n}(\psi) = (P, f).$$

Proof: Let a_i be the singular values of (P, f). Put

$$\widetilde{P} = P \setminus \cup f^{-1}(a_i),$$

$$\widetilde{\mathbb{C}} = \bar{\mathbb{C}} \setminus \cup a_i,$$

and

$$\widetilde{f} = f \mid_{\widetilde{P}}.$$

Then

$$\widetilde{f} : \widetilde{P} \to \widetilde{\mathbb{C}}$$

is a covering without singular points. From Lemma 2 it follows that there exists a $\psi \in T_k$ and a covering

$$\varphi_\Lambda : \Lambda \to \widetilde{C} = \Lambda/\psi(S_k).$$

Thus there exists

$$\widetilde{\Gamma} \subset \psi(S_k)$$

such that

$$\varphi_\Lambda = \widetilde{f}\widetilde{\varphi}_\Lambda,$$

where

$$\widetilde{\varphi}_\Lambda : \Lambda \to \widetilde{P} = \Lambda/\widetilde{\Gamma}$$

is the natural projection. Put

$$S^n = \psi^{-1}(\widetilde{\Gamma}) \subset S_k.$$

Then

$$\widetilde{\Gamma} = \psi(S^n)$$

and

$$\Psi_{S^n}(\psi) = (P, f).$$

<u>Lemma 4</u>. *Suppose that the meromorphic functions (P, f) and (P', f') have the same topological type and*

$$(P, f) = \Psi_{S^n}(\psi),$$

where

$$S^n \subset S_k, \quad \psi \in T_k.$$

Then there exists a $\psi' \in T_k$ such that

$$(P', f') = \Psi_{S^n}(\psi').$$

<u>Proof:</u> From Lemma 3 it follows that there exists $S_1^n \subset S_k$ and $\psi_1' \in T_k$ such that

$$(P', f') = \Psi_{S_1^n}(\psi_1').$$

Put

$$\widetilde{P} = \Lambda/\psi(S^n), \quad \widetilde{P}' = \Lambda/\psi_1'(S_1^n),$$

$$\widetilde{C} = \Lambda/\psi(S_k), \quad \widetilde{C}' = \Lambda/\psi_1'(S_k).$$

Since (P, f) and (P', f') have the same topological type, there exists a homeomorphism

$$\varphi_\Lambda : \Lambda \to \Lambda$$

such that

$$\psi_1'(S_k) = \varphi_\Lambda \psi(S_k) \varphi_\Lambda^{-1}$$

and

$$\psi_1'(S_1^n) = \varphi_\Lambda \psi(S^n) \varphi_\Lambda^{-1}.$$

The homeomorphism φ_Λ gives rise to the isomorphism

$$\varphi_\psi : \psi(S_k) \to \psi'(S_k),$$

where

$$\varphi_\psi(\psi(s)) = \varphi_\Lambda \psi(s) \varphi_\Lambda^{-1}$$

for $s \in S_k$. Put

$$h = (\psi_1')^{-1} \varphi_\psi \psi : S_k \to S_k$$

and

$$\psi' = \psi_1' h.$$

Thus

$$h(S^n) = S_1^n$$

and

$$(P', f') = \Psi_{S_1^n}(\psi_1') = \Psi_{h(S^n)}(\psi_1' h) = \Psi_{S^n}(\psi').$$

Theorem 3. *Let (P, f) be a meromorphic function with k singular values . Let H be the space of all meromorphic functions of the same topological type as (P, f). Then*

$$H = T_k / Mod \cong \mathbb{R}^{2k-6} / Mod,$$

where

$$Mod \subset Mod_k.$$

Moreover, if $(P, f) = \Psi_S(\psi)$ then

$$Mod = \{h \in Mod_k \mid h(S) = S\}.$$

Proof: From lemma 3 it follows that there exists $S \subset S_k$ such that

$$(P, f) = \Psi_S(\psi).$$

From Lemma 4 it follows that

$$H \subset \Psi_S(T_k).$$

From Theorem 1 it follows that T_k is a connected space. Moreover, from the description of Ψ_S it follows that functions $\Psi_S(\psi_1)$ and $\Psi_S(\psi_2)$ have the same topological type if ψ_1 and ψ_2 are near each other. Thus

$$\Psi_S(T_k) \subset H$$

and

$$H = \Psi_S(T_k).$$

Moreover

$$\Psi_S(T_k) = T_k / \text{Mod},$$

where

$$\text{Mod} = \{\alpha \in \text{Mod} \mid \alpha S = \beta S \beta^{-1}, \beta \in S_k\}.$$

Thus, according to Theorem 1,

$$H = T_k/\mathrm{Mod} \cong \mathbb{R}^{2k-6}/\mathrm{Mod}.$$

Corollary There exists a canonical covering with a finite number of sheets

$$H \to M_k.$$

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