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P. A. ZALESSKII

VIRTUALLY FREE PRO-P GROUPS

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P. A. Zalesskii*

In this paper we overview recent results on virtually free pro-p groups, i.e., pro-p groups having open free subgroups. We describe also applications of these results to the study of automorphisms of finite order of free pro-p groups.

The first result on virtually free pro-p groups was obtained by Serre in a seminal paper [Serre 65]:

Theorem 1 . [Serre 1965] *Let G be a pro-p-group and F an open free pro-p subgroup of G. If G is torsion free, then G is also a free pro-p group.*

This led Serre to conjecture that a (discrete) torsion free virtually free group is free. With this conjecture he essentially initiated the study of groups of cohomological dimension 1 in the discrete case. He showed that any such group has cohomological dimension at most 1 over **Z**. Subsequently, [Stallings 1968] proved that every finitely generated group of cohomological dimension 1 over **Z** is free, and then [Swan 1969] eliminated the assumption of finite generation, thus establishing the conjecture in full generality.

Karrass, Pietrowski, Solitar, Cohen and Scott progressively extended the above result culminating in the following characterization of virtually free groups.

Theorem 2 . [Karrass-Pietrowski-Solitar 73], [Cohen 73], [Scott 74]. *A group G is virtually free if and only if it is the fundamental group of a finite graph of finite groups.*

Karrass, Pietrowski and Solitar proved the above theorem in the finitely generated case, Cohen in the countably generated case, and Scott in general.

In the special case where G has a free subgroup of prime index, Dyer and Scott gave the following description of G as a free product.

Theorem 3 . [Dyer-Scott 75]. *Let G be a group having a free subgroup of index p. Then $G = (*_{i\in I}(C_p \times H_i)) * H$, where H_i, H are free groups.*

As one can see, the whole development began with the study of pro-p groups. However, the results in the discrete situation are at present much stronger. This can be exlpained by the fact that the proof of the discrete result is based on Stallings' theory of ends and on Bass-Serre's theory of groups acting on trees, and there is no pro-p analogue of theory of ends (and it is not clear whether it is possible to develop such an analogue). The pro-p version of the theory of groups acting on trees was developed in [Gildenhuys-Ribes 78], [Zalesskii-Melnikov (1) 89], [Zalesskii 89] and [Zalesskii-Melnikov (2) 89], and these results provided the context for formulating an appropriate pro-p version of the Karrass-Pietrovski-Solitar-Cohen-Scott result as a conjecture. Even a pro-p version of Dyer-Scott's theorem could have been conjectured only after [Haran 87] and [Melnikov 89] provided a sufficiently general definition of a free pro-p product.

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Conjecture 1 Let G be a virtually free pro-p group. Then G is the fundamental group of a profinite graph of finite p-groups of bounded order in the category of pro-p groups.

Conjecture 2 Let G be a pro-p group having a free subgroup of index p. Then $G = (\coprod_{x\in X}(C_p \times H_x)) \amalg H$ is a free pro-p product, where H_x, H are free pro-p groups.

Some of the interest in this topic comes from number theory; for instance, virtually free pro-2 groups have been studied in the contex of Galois theory in [Haran 93], [Engler 95].

The strategy for proving these conjectures is to use (co)homological methods and the pro-p analogue of the Bass-Serre theory of groups acting on trees. In fact, Serre used cohomological methods for proving Theorem 1, deducing it from the following result

Theorem 4 . [Serre 65] *Let G be a torsion free profinite group of virtual cohomological dimension* $\text{vcd}G = n < \infty$. *Then G has cohomological dimension* $\text{cd}G = n$.

Recently C.Scheiderer [Scheiderer 98] gave a very elegant proof of the pro-p analogue of Dyer and Scott theorem in the finitely generated case. As the main ingredient of this proof Scheiderer used a generalization of the above theorem of Serre (Theorem 4), describing the higher-dimensional cohomology of certain profinite groups of finite virtual cohomological dimension in terms of the cohomology of their finite subgroups (see [Scheiderer 94]). We state here a homological finitely generated pro-p version of this result.

Theorem 5 . [Scheiderer 94] *Let G be a finitely generated pro-p group of virtual cohomological dimension d which does not contain any subgroup isomorphic to $C_p \times C_p$. Then for any $n > d$, the homology group $H_n(G, \mathbf{Z}/p\mathbf{Z})$ is isomorphic to a topological direct sum*

$$H_n(G, \mathbf{Z}/p\mathbf{Z}) \cong \oplus_T H_n(T, \mathbf{Z}/p\mathbf{Z}),$$

where T ranges over a system of representitives of the conjugacy classes of subgroups of G of order p. In fact, the isomorphism is induced from the corestriction maps $H_n(T, \mathbf{Z}/p\mathbf{Z}) \hookrightarrow H_n(G, \mathbf{Z}/p\mathbf{Z})$.

The Scheiderer pro-p analogue of the Dyer-Scott theorem was extended by W.Herfort, L.Ribes and P. Zalesskii [Herfort-Ribes-Zalesskii 98] to the general case, thus proving Conjecture 2:

Theorem 6 . [Scheiderer 98], [Herfort-Ribes-Zalesskii 98]. *If G is a pro-p group having a free pro-p subgroup F of index p then*

$$G \cong (\coprod_{x\in X}(C_p \times H_x)) \coprod H,$$

is a free pro-p product, where C_p denote the group of order p, H_x, H are free pro-p subgroups of F and X is the space of conjugacy classes of subgroups of order p in G.

Conjecture 1 has recently been established in [Herfort-Zalesskii 99] for cyclic extensions of free pro-p groups.

Theorem 7 . [Herfort-Zalesskii 99] *Let G be a cyclic extension of a free pro-p group. Then G is the fundamental group of a profinite graph of finite p groups of bounded order.*

Furthermore, free-by-cyclic pro-p groups admit an internal description as a free product along the lines of Theorem 6.

Theorem 8 . [Herfort-Zalesskii 99] *Let G be a cyclic extension of a free pro-p group F. Then*

$$G \cong \coprod_{x \in X} N_G(C_x) \amalg H,$$

is a free pro-p product, where $\{C_x \mid x \in X\}$ is a system of representatives of conjugacy classes of subgroups of order p in G, $N_G(C_x)$ is the normalizer of C_x in G, and H is a free pro-p subgroup of F.

It turns out that Theorem 7 is the best possible result one can obtain without restrictions on the rank of F. In [Herfort-Zalesskii 99] an example of a semidirect product $F \rtimes (C_2 \times C_2)$ which can not be represented as the fundamental group of a profinite graph of finite p groups is given, showing that Conjecture 1 does not hold in general. The reason for this seems to be purely topological, arising from the fact that for a pro-p group acting on a profinite space X of large cardinality (for example $\aleph_2$) the quotient map $X \longrightarrow X/G$ does not always admit a continuous section (retract). It is reasonable to believe that Conjecture 1 is likely to be true if the rank of the free subgroup F is not so large, in particular, when it is finite. The following result provides strong support for this.

Theorem 9 .. [Herfort-Ribes-Zalesskii (1) 98] *Let G be a finite extension of free pro-p group of $rank(F) < 3$. Then G is the fundamental pro-p group of a finite graph of finite p-groups.*

In fact, the information obtained in [Herfort-Ribes-Zalesskii (1) 98] is very precise. For example the next theorem gives a very explicit description of finite extensions of a free group of rank 2 having trivial center.

Theorem 10 . [Herfort, Ribes, Zalesskii (1) 98] *Let G be a pro-p-group with trivial center having an open normal free subgroup F of rank 2. Then G has one of the following structures:*

1) G is a free pro-p group of finite rank.

2) $p = 3$ and $G \simeq C_3 \coprod C_3$

3) $p = 2$ and G has one of the following forms:

a) $G \simeq C_2 \coprod C_2 \coprod C_2$;

b) $G \simeq C_2 \coprod \mathbf{Z}_2$;

c) $G \simeq C_2 \coprod (C_2 \times \mathbf{Z}_2)$;

d) $G \simeq C_4 \coprod C_2$;

e) $G \simeq (C_2 \times C_2) \coprod C_2$;

f) $G \simeq (\mathbf{Z}_2 \times C_2) \coprod_{C_2} (C_2 \times C_2) \coprod_{C_2} (C_2 \times \mathbf{Z}_2)$

We now turn to automorphisms of free pro-p groups. It is well-known that the automorphism group $\mathrm{Aut}(F)$ of a free pro-p group of finite rank is a profinite group, in fact a virtually pro-p group, i.e. $\mathrm{Aut}(F)$ contains an open pro-p subgroup. It follows that one can consider the order of an automorphism of F as supernatural number rp^t, where r is a natural number relatively prime to p and $0 \leq t \leq \infty$.

By a celebrated theorem of Gersten the group of fixed points of an automorphism of a free abstract group of finite rank is finitely generated. The next theorem shows that the pro-p analogue of this theorem does not hold, unless one makes some restrictions on the order of the automorphism.

Theorem 11 . [Herfort-Ribes 90] *Let F be a free pro-p group of rank$(F) = n > 1$ and α an automorphism of F. If the order m of α is coprime to p, then the rank of the group of fixed points $Fix_F(\alpha)$ is necessarily infinite.*

This result is less surprising if one considers the holomorph $F \rtimes \langle \alpha \rangle$, and $\mathrm{Fix}_F(\alpha)$ as the centralizer of α. Indeed, the properties of virtually free pro-p groups are in many cases similar to those of abstract free groups, but may change radically if one is at some point forced to go outside the class of pro-p groups. Thus, one might resonably to conjecture that the appropriate pro-p analogue of Gersten's theorem will require in addition that the holomorph be a pro-p group, or, equivalently, that the order of α be p^t for $0 \leq t \leq \infty$.

Conjecture 3 [Herfort-Ribes-Zalesskii 95] Let F be a free pro-p group of rank n, and α an automorphism of F of order p^t $(0 \leq t \leq \infty)$. Then the rank of $\mathrm{Fix}_F(\alpha)$ is at most n.

If the order of the automorphism α is finite, an affirmative answer to Conjecture 3 follows from Theorem 6. Indeed, in this case the holomorph $F \rtimes \langle \alpha \rangle$ is a virtually free group, and its subgroup $F \rtimes \langle \alpha^{p^{t-1}} \rangle$ satisfies the assumptions of Theorem 6. Hence by applying that theorem subsequently t times in success (or using induction on t) one deduces the following result.

Theorem 12 . [Scheiderer 98], [Herfort-Ribes-Zalesskii 98] *Suppose F is a free pro-p group and α is an automorphism of F of order p^t $(t < \infty)$. Then the set of fixed points $Fix_F(\alpha)$ is a free factor of F. In particular, rank*$(\mathrm{Fix}_F(\alpha))$ $\leq$*rank(F).*

It is known that the automorphism group $\mathrm{Aut}(F_n)$ of a free pro-p group of rank $n > 1$ is much more complicate than the automorphism group $\mathrm{Aut}(\Phi_n)$ of the abstract free group Φ_n of rank n. Athough $\mathrm{Aut}(\Phi_n)$ is embedded in $\mathrm{Aut}(F_n)$, it is by no means dense there. In fact, V.Romankov has proved that $\mathrm{Aut}(F_n), n > 1$, is (topologically) infinitely generated! (See [Romankov 96]). Nevertheless, Theorem 8 affords a description of the conjugacy classes of the automorphisms of finite order of a free pro-p group.

Theorem 13 . [Herfort-Zalesskii 99] *Let F be a free pro-p group.*

(1) There exists a dense abstract free subgroup Φ of the same rank as F, such that each conjugacy class of automorphisms of order p^n in $Aut(F)$ intersects precisely one conjugacy class of automorphisms of order p^n in $Aut(\Phi)$;

(2) The conjugacy classes of automorphisms of F having order q coprime to p are in one-to-one correspondence to the conjugacy classes of automorphisms of order q of the Frattini quotient F/F^*.

An application of Theorem 10 gives a precise list of the number of conjugacy classes of a given order automorphism of a free pro-p group of rank 2.

Theorem 14 . [Herfort, Ribes, Zalesskii (1) 98] *Let S denote the set of all possible orders of torsion elements of the automorphism group $\mathrm{Aut}(F_2)$ of a free pro-p group of rank 2. Let $c(s)$ is the number of conjugacy classes of automorphisms of order s. Then*

(1) if s is coprime to p, then $s \mid p^2 - 1$, and one of the following holds:

(a) $s \mid p-1$ and $c(s) = \phi(s)s$

(b) $s \not| p-1$ and $c(s) = \frac{\phi(s)}{2}$,

where ϕ denotes Euler's function;

(2) if $p = 2$, then $S = \{2,3,4\}$ and $c(2) = 4$, $c(3) = 1$, $c(4) = 1$;

(3) if $p = 3$, then $S = \{2,3,4,8\}$ and $c(2) = 2$, $c(3) = 1$, $c(4) = 1$, $c(8) = 2$;

(4) if $p > 3$, then any $s \in S$ is coprime to p, so that the formulas in (1) hold.

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address

Institut für Angewandte und Numerische Mathematik
Technische Universität Wien
Wiedner Hauptstr. 8-10/1151
A-1040 Wien
Austria