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RIGID ISOTOPIES OF THE REAL PROJECTIVE CONFIGURATIONS

Rigid isotopies of the real projective configurations

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1. The main objects. An *ordered (unordered) (n;k)-configuration of degree m* is defined to be an ordered (respectively unordered) collection of m linear k-dimensional subspaces of $\mathbb{R}P^n$. We associate with each configuration its *upper* and *lower ranks*, i.e. the dimensions of the projective hull and intersection respectively of all the subspaces of the configuration. The *combinatorial characteristic* of a configuration is, by definition, the list of upper and lower ranks of all its subconfigurations. Two configurations are said to be *rigidly isotopic* if they can be joined by isotopy consisting of configurations with the same combinatorial characteristics. It is obvious that the property of being rigidly isotopic is equivalence relation. The equivalence class of a configuration with respect to this relation is called its *rigid isotopy type*.

The space $PC^m_{n,k}$ ($SPC^m_{n,k}$) of ordered (unordered) (n;k)-configurations of degree m is naturally isomorphic to the m-th (symmetric) power of the Grassmanian $G_{n+1,k+1}$. A configuration is said to be *non-singular* if all its subspaces are in general position. The set $GPC^m_{n,k}$ ($GSPC^m_{n,k}$) of non-singular ordered (unordered) configurations is an open subset of $PC^m_{n,k}$ ($SPC^m_{n,k}$) in Zariski topology. The set of all non-singular ordered (unordered) configurations of the same rigid isotopy type forms a connected component of $GPC^m_{n,k}$ ($GSPC^m_{n,k}$) in strong topology. These connected components are called *cameras* of $PC^m_{n,k}$ ($SPC^m_{n,k}$).

2. Adjacency graph. A configuration is said to be 1-*singular* if all configurations rigidly isotopic to it form a codimension 1 subset in the configuration space. The set of all 1-singular configurations of the same rigid isotopy type is called a *wall*. Two 1-singular configurations are said to be *p-equivalent* if they belong to walls which separate the same cameras.

The mutual position of the cameras in the configuration space can be described by means of the *adjacency graph* (see [2]), whose vertices and edges are in one-to-one correspondence with the cameras and walls respectively, and two vertices representing some cameras are connected by an edge if and only if these cameras are adjacent to the wall corresponding to this edge. It may happen that the end points of an edge coincides with each other, as in the following cases:

a) if the configuration space has a boundary and the wall is contained in it;

b) if the wall is a one-sided subset of the configuration space;

c) if the wall is a two-sided subset, but has the same camera adjacent at each side.

Each of these cases corresponds to a loop in the adjacency graph. In cases b) and c) the wall is called *inner*.

3. Degeneration and perturbation. Let X be $PC^m_{n,k}$ or $SPC^m_{n,k}$, $A \in X$, $s\colon [0,1] \longrightarrow X$ be a path such that $A = s(0)$ and the restriction $s|_{[0,1)}$ is a rigid isotopy of A. If configurations A and $A' = s(1)$ have distinct combinatorial characteristics, then s is called a *degeneration* of configuration A, and path s^{-1} is called a *perturbation* of A'.

Let $A = \{A_1, \dots, A_m\}$ be a non-singular configuration of k-dimensional subspaces of $\mathbb{R}P^{2k+1}$. We say that the subspaces A_i and A_j of A can be *moved up to intersection in a point*, if there exists a degeneration of A such that:

1) its restrictions to the subconfigurations $\{A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_m\}$ and $\{A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_m\}$ which are obtained by removing the elements A_i, A_j from A respectively are rigid isotopies,

2) in the result of this degeneration the subspaces corresponding to the subspaces A_i and A_j intersect in a point.

4. Linking numbers. In the next two sections I describe the constructions of O.Ya.Viro (see [9], [10]).

Let $B=\{B_1, B_2, B_3\}$ be ordered non-singular configuration of three k-dimensional subspaces of oriented space $\mathbb{R}P^{2k+1}$. Consider the canonical projection pr: $\mathbb{R}^{2k+2}\setminus\{0\} \longrightarrow \mathbb{R}P^{2k+1}$. The orientation of $\mathbb{R}P^{2k+1}$ induces the orientation of vector space $\mathbb{R}^{2k+2}$. Let $\bar{B}_i = pr^{-1}(B_i) \cup \{0\}$, i=1,2,3. It is clear that $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$ are vector (k+1)-dimensional subspaces of $\mathbb{R}^{2k+2}$. Let $\bar{B}^*_1$, $\bar{B}^*_2$, $\bar{B}^*_3$ be the same subspaces equiped with some orientations. To every ordered pair $(\bar{B}^*_i, \bar{B}^*_j)$, where i,j = 1,2,3, $i\neq j$, we assign a integer denoted by $lk(\bar{B}^*_i,\bar{B}^*_j)$ which is equal to +1 if the orientation of $\mathbb{R}^{2k+2}$ coincides with the orientation induced by $\bar{B}^*_i\oplus\bar{B}^*_j$, and equal to -1 in the opposite case. The product $lk(\bar{B}^*_2,\bar{B}^*_3)lk(\bar{B}^*_3,\bar{B}^*_1)lk(\bar{B}^*_1,\bar{B}^*_2)$ denoted by $lk(B_1,B_2,B_3)$ is called the *linking number* of the triple of disjoint k-dimensional subspaces B_1, B_2, B_3 in the oriented space $\mathbb{R}P^{2k+1}$. It is easy to see that $lk(B_1,B_2,B_3)$ does not depend on the choice of the orientations of $\bar{B}_i$, i = 1,2,3, is preserved under rigid isotopies of B, and changed under reversal of the orientation of $\mathbb{R}P^{2k+1}$. It can be observed also that if k is odd, then $lk(B_1,B_2,B_3)$ does not depend on the choice of orders of subspaces of B and is preserved under isotopies of B (in this case $lk(\bar{B}^*_i,\bar{B}^*_j)$ coincides with the doubled linking number of cycles B^*_i and B^*_j in the oriented manifold $\mathbb{R}P^{2k+1}$).

Ordered non-singular (2k+1;k)-configurations $C = \{C_1, C_2, \dots, C_m\}$ and $C' = \{C'_1, C'_2, \dots, C'_m\}$ are said to be *homology equivalent* if for a fixed orientation of $\mathbb{R}P^{2k+1}$ $lk(C_i,C_j,C_k) = lk(C'_i,C'_j,C'_k)$ for any i,j,k = 1,2,...,m, i<j<k. Two ordered 1-singular (2k+1;k)-configurations will be called *homology equivalent* if after perturbations of them two pairs of homology equivalent ordered non-singular configurations are obtained. And two unordered 1-singular (non-singular) (2k+1;k)-configurations will be called *homology equivalent* if they can be ordered by the such way that the ordered configurations corresponding to them are homology equivalent.

5. Join summation and suspension. Let $A = \{A_1, \dots, A_m\}$ be an ordered configuration of k-dimensional subspaces of $\mathbb{R}P^n$, and let $B = \{B_1, \dots, B_m\}$ be an ordered configuration of l-dimensional subspaces of $\mathbb{R}P^s$. We suppose that $\mathbb{R}P^n$ and $\mathbb{R}P^s$ are imbedded into $\mathbb{R}P^{n+s+1}$ as disjoint linear subspaces. If n and s are odd, we suppose, in addition, that $\mathbb{R}P^n$, $\mathbb{R}P^s$, and $\mathbb{R}P^{n+s+1}$ are oriented, and linking number of the images of $\mathbb{R}P^n$ and $\mathbb{R}P^s$ in $\mathbb{R}P^{n+s+1}$ equals +1. Let C_i be the projective hull of the images of A_i and B_i in $\mathbb{R}P^{n+s+1}$, i= 1,...,m. It is clear that $C = \{C_1, \dots, C_m\}$ is an ordered $(n+s+1;k+l+1)$-configuration of degree m. The configuration C is called the join of A and B. A configuration is called an isotopy join if it is rigidly isotopic to the join of some two configurations.

In the same manner, one can determine the join sum of two unordered configurations, but in this case it depends on the choice of orders of the elements of the configurations summarized.

A non-singular ordered (unordered) (3;1)-configuration which consists of generatrices of a quadric in $\mathbb{R}P^3$ will be called an ordered (unordered) *trivial configuration* of lines of $\mathbb{R}P^3$. A trivial configuration of lines of oriented space $\mathbb{R}P^3$ with positive linking numbers of triples of the lines will be called the *Hopf configuration.* The join of a configuration of k-dimensional subspaces of $\mathbb{R}P^n$ and the trivial configuration (the Hopf configuration, if n is odd) is called the *suspension* of this (n;k)-configuration.

It is easy to see that any two lines of a trivial configuration can be transposed by a rigid autoisotopy of the configuration which keeps other lines of the configuration fixed. It follows that one can find a rigid autoisotopy of this configuration which permutes its lines in an arbitrary way. Hence, the join of an unordered configuration of k-dimensional subspaces of $\mathbb{R}P^n$ and the trivial configuration (the Hopf configuration, if n is odd) does not depend on the orders of elements of these configurations up to rigid isotopy.

5.1. **Lemma.** The construction of the suspension preserves the linking numbers and rigidly isotopic configurations are taken to rigidly isotopic configurations.

6. Configurations of at most 6 lines of $\mathbb{R}P^3$. Let L^1 and L^2 be two oriented disjoint lines in $\mathbb{R}P^3$ with positive linking number. Let $A^1_1, \ldots,$ A^1_m be some different points of L^1, and $A^2_1, \ldots, A^2_r$ be some different points of L^2 such that the order of the points determined by the lower indices agrees to the orientations of L^1 and L^2. We suppose that $r \leq m$. Consider a map f from $\{1, \ldots m\}$ onto $\{1, \ldots, r\}$ and connect points A^1_i and $A^2_{f(i)}$ by lines, $i = 1, \ldots, m$. If $r = m$ we obtain a non-singular join (3;1)-configuration of degree m which is denoted by jc(f); if $r = m-1$, we obtain 1-singular join (3;1)-configuration of degree m which is denoted by sjc(f). Since f can be represented by the table $\begin{bmatrix} 1 & \ldots & m \\ f(1) & \ldots & f(m) \end{bmatrix}$ we use the symbol $(f(1), \ldots, f(m))$ to denote f.

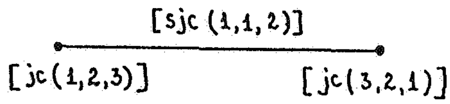
6.1. **Theorem.** (O.Ya.Viro, see [9]). Any unordered non-singular configuration of at most 5 lines of $\mathbb{R}P^3$ is an isotopy join. Two unordered non-singular (3;1)-configurations of degree ≤ 5 are rigidly isotopic if and only if they are homology equivalent.

6.2. **Theorem.** Any unordered 1-singular configuration of at most 5 lines of $\mathbb{R}P^3$ is an isotopy join. Two unordered 1-singular (3;1)-configurations of degree ≤ 5 are rigidly isotopic if and only if they are homology equivalent.

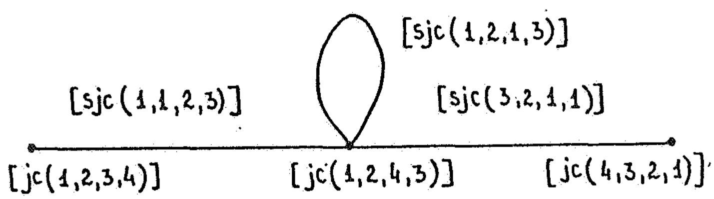
We shall denote the rigid isotopy type of configuration K by [K]. The next theorem follows from Theorems 6.1 and 6.2.

6.3. **Theorem.** a) The adjacency graph of $SPC^2_{3,1}$ has one vertex and one edge-loop, which corresponds to the one-sided inner wall.

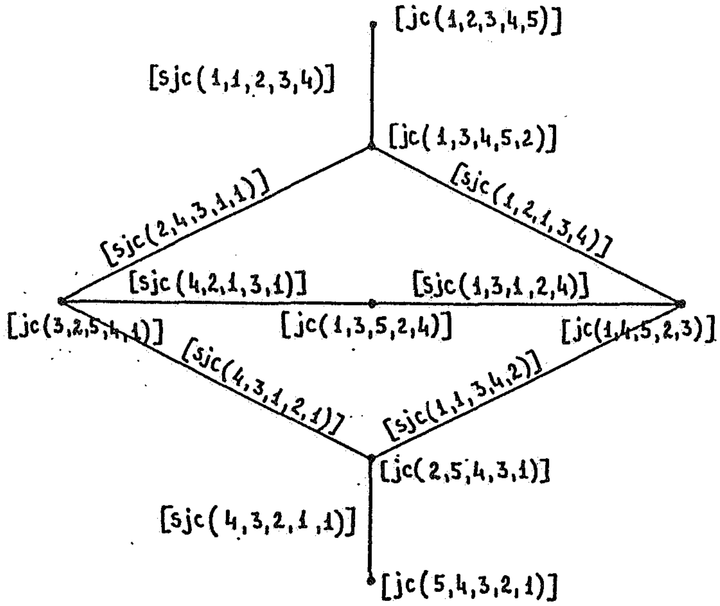
b) The adjacency graph of $SPC^3_{3,1}$ is shown on the diagram below.



c) The adjacency graph of $SPC^4_{3,1}$ is the graph presented below. The loop corresponds to one-sided inner wall.



d) The adjacency graph of $SPC^{5}_{3,1}$ is the following.



Unlike the non-singular (3;1)-configurations of degree ≤ 5 unordered non-singular configurations of 6 lines of $\mathbb{R}P^3$ are not determined up to rigid isotopy by the linking numbers.

6.4. **Theorem.** a) The unordered non-singular (3;1)-configuration M, affine part of which is shown on diagram 1 in Appendix, and its mirror image are homology equivalent, but not rigidly isotopic.

b) The unordered non-singular (3;1)-configuration L, affine part of which is shown on diagram 2 in Appendix, and unordered non-singular join (3;1)-configuration jc(1,2,5,6,3,4) are homology equivalent, but not rigidly isotopic.

c) The mirror images of L and jc(1,2,5,6,3,4) are homology equivalent, but not rigidly isotopic.

We denote the mirror image of the configurations L and M by L′ and M′

respectively.

Yu.V.Drobotukhina in [1] defined an analogous of Jones polynomial for links in $\mathbb{R}P^3$. This polynomial was defined by means of the state model analogous to Kauffman's model [3] for the Jones polynomial of links in S^3. This construction yields the bracketed Kauffman polynomial of links in $\mathbb{R}P^3$. This polynomial is not an isotopy invariant of links in $\mathbb{R}P^3$, since it is not preserved under the Reidemeister motion Ω_1 of links diagram. Nevertheless, it is an rigid isotopy invariant of unordered non-singular configurations of projective lines, since in the process of rigid isotopy this Reidemeister motion does not occur. This polynomial will be called the *Kauffman polynomial* of a non-singular configuration of lines of $\mathbb{R}P^3$.

6.5. **Theorem.** Any unordered non-singular (3;1)-configuration of degree 6 is either an isotopy join, or rigidly isotopic to one of the following four pairwise non-isotopic configurations: L, M, L′, M′. Two unordered non-singular configurations of 6 lines of $\mathbb{R}P^3$ are rigidly isotopic if and only if their Kauffman polynomials are equal.

6.6. **Theorem.** Any unordered 1-singular (3;1)-configuration of degree 6 is either an isotopy join, or rigidly isotopic to one of the following six pairwise non-isotopic configurations: $\mathfrak{K}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{K}'$, $\mathfrak{L}'$, $\mathfrak{M}'$. Two unordered 1-singular configurations of 6 lines of $\mathbb{R}P^3$ are rigidly if and only if they are homology equivalent and p-equivalent simultaneously.

The line of a non-singular join (3;1)-configuration $jc(\tau_1, \ldots ,\tau_m)$ corresponding to the element τ_p (where $p \in \{1, \ldots ,m\}$) will be denoted by symbol (τ_p). The 1-singular (3;1)-configurations $\mathfrak{K}$, $\mathfrak{L}$, $\mathfrak{M}$ are obtained from the non-singular (3;1)-configurations jc(1,3,5,2,6,4), L, M by moving up to the intersection in a point the lines (1) and (5), L_2 and L_4, M_2 and M_4 respectively. The 1-singular configurations $\mathfrak{K}'$, $\mathfrak{L}'$, and $\mathfrak{M}'$ are the mirror images of $\mathfrak{K}$, $\mathfrak{L}$, and $\mathfrak{M}$ respectively.

The next theorem is a consequence of Theorems 6.5 and 6.6.

6.7. **Theorem.** The adjacency graph of $SPC^6_{3,1}$ is shown on diagram 3 in Appendix. All loops of the graph correspond to one-sided inner walls.

7. Stabilization properties of the suspension. O.Ya.Viro put forward two following hypothesis (The Bielefeld meeting on combinatorial theory, 1989):

1) to some extent construction of the suspension realized an embedding of the theory of the rigid isotopy types of (2k+1;k)-configurations into the theory of the rigid isotopy types of (2k+5;k+2)-configurations;

2) there exists an analogous of the Kauffman polynomial for non-singular (4n-1;2n-1)-configurations and it is preserved under the suspension.

In 1990 I showed that the first Viro's hypothesis was true. Namely I proved the following theorem.

7.1. **Theorem.** Any 1-singular (non-singular) (2k+5;k+2)-configuration of degree m is rigidly isotopic to the suspension of a 1-singular (non-singular) (2k+1;k)-configuration for $m \le k+5$ $(k > 0)$, and, if $m \le k+2$, then rigid isotopy of the suspensions is equivalent to rigid isotopy of the original (2k+1;k)-configurations.

In 1990 I also showed that the second Viro's hypothesis is not true as a whole. Namely the following theorem is true.

7.2. **Theorem.** Three pairs of unordered non-singular (3;1)-configurations of degree 6 from Theorem 6.4 have distinct Kauffman polynomials, but the suspensions of these configurations are rigidly isotopic.

8. Configurations of at most six (2n-1)-dimensional subspaces of $\mathbb{R}P^{4n-1}$.

8.1. **Theorem.** For n>1 two unordered 1-singular (non-singular) (4n-1;2n-1)-configurations of degree ≤ 6 are rigidly isotopic if and only if they are homology equivalent.

8.2. **Theorem.** The adjacency graphs of $SPC^{m}_{4n-1,2n-1}$ and $SPC^{m}_{3,1}$ are isomorphic to one another for any n>1 and $2\leq m\leq 5$. If n>1, the adjacency graph of $SPC^{6}_{4n-1,2n-1}$ coincides with Diagram 4 in Appendix. All the loops of these graphs correspond to one-sided inner walls.

9. Stable equivalence of real projective configurations. We say that two (n;k) configurations of degree m are stable equivalent if their s-fold suspensions are rigidly isotopic for some s.

The following theorem was proved by me and S.Hashin independently.

9.1. **Theorem.** Two ordered (unordered) non-singular (4n-1;2n-1)-configurations are stable equivalent if and only if they are homology equivalent.

9.2. **Remark.** The analogous result is true for the 1-singular configurations.

10. Isotopy join configurations of lines of $\mathbb{R}P^3$. The following theorem was shown by my and S.Hashin together.

10.1. **Theorem.** Two ordered (unordered) non-singular isotopy join (3;1)-configurations are rigidly isotopic if and only if they are homology equivalent.

10.2. **Remark.** The analogous result is true for the 1-singular configurations.

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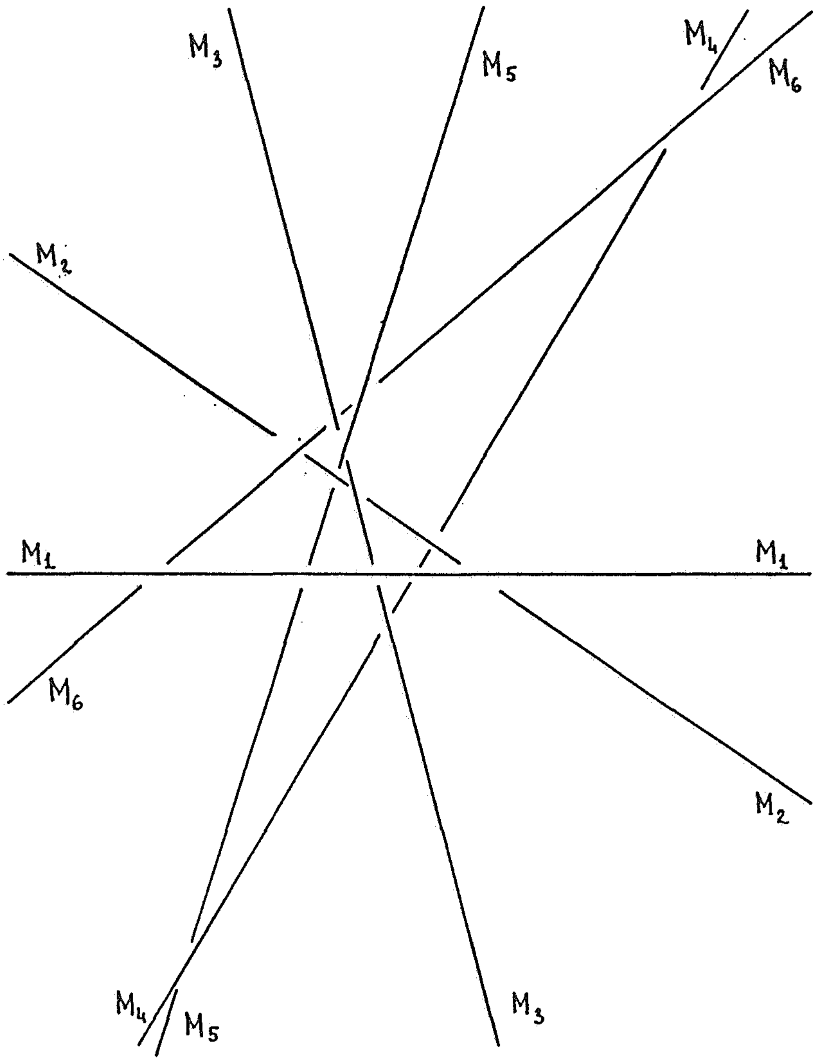
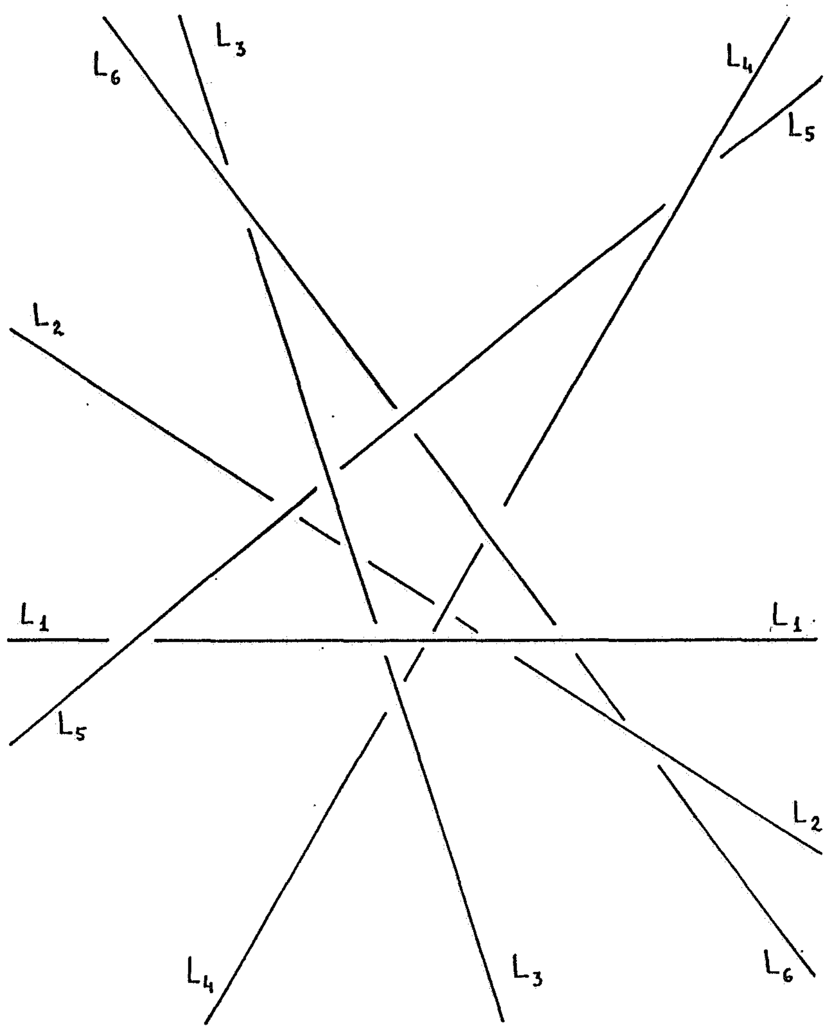
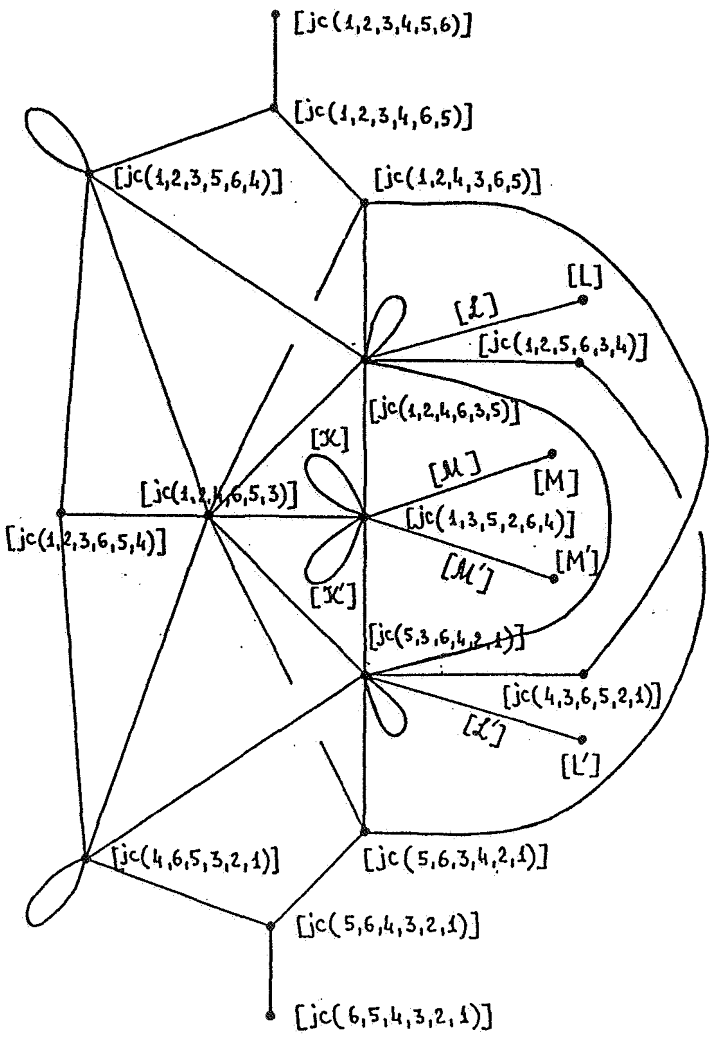


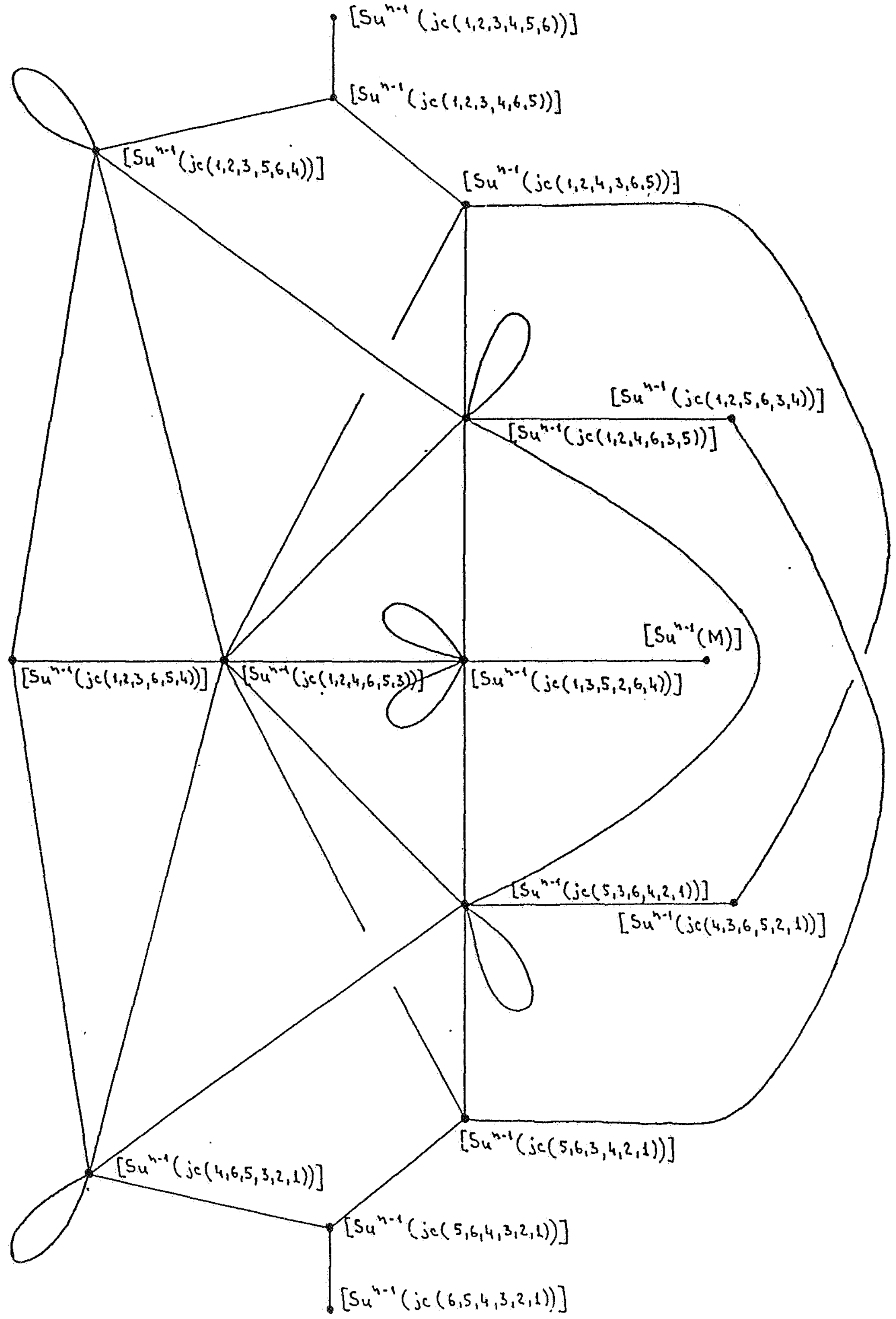
DIA. 1. Affine part of non-singular (3; 1)-configuration M.



DIA. 2. Affine part of non-singular (3; 1)-configuration L.



DIA. 3. Adjacency graph of $SPC^{6}_{3,1}$.



DIA. 4. Adjacency graph of $SPC^{6}_{4n-1,2n-1}$ for $n>1$.