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GENERALIZED LINS-MANDEL SPACES AND BRANCHED COVERINGS OF S^3

GENERALIZED LINS-MANDEL SPACES AND BRANCHED COVERINGS OF $\mathbb{S}^3$

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It is well known that PL-manifolds are representable by edge-coloured graphs. This combinatorial approach turns out to be a useful tool for generating and investigating wide classes of manifolds represented by "highly-symmetric" graphs (see, for example, [6], [11], [14] and [46]). In particular, Lins-Mandel spaces ([29]) have been intensively studied in the last ten years. In this paper, after a short survey on the related research areas, we describe recent results (mainly obtained in the Ph.D. thesis [40] of one of the authors) about the topological structure of Lins-Mandel spaces in terms of branched coverings of $\mathbb{S}^3$. We also illustrate generalizations of these spaces and their relations with other representation theories.

1. BRANCHED COVERINGS OF $\mathbb{S}^3$

With the term *manifold* we always mean a compact, connected, orientable PL-manifold without boundary.

Let $\tilde{M}, M$ be triangulated n-manifolds and let N be an $(n-2)$-subcomplex of M; a non-degenerate map $f : \tilde{M} \to M$ is a *d-fold covering map, branched over* N, if:

- $f' = f_{|\tilde{M}-f^{-1}(N)}$ is an ordinary covering of $M - N$ of degree d;
- $N = \{x \in M \mid \#f^{-1}(x) < d\}$ (*branching set*).

$\tilde{M}$ is said to be a *branched covering* of M.

$$\begin{array}{ccc} \tilde{M} - f^{-1}(N) & \xrightarrow{i} & \tilde{M} \\ f' \downarrow & & \downarrow f \\ M - N & \xrightarrow{i} & M \end{array}$$

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A remarkable result by R.H.Fox ([18]) states that a branched covering is uniquely determined by the ordinary covering induced by restriction. This proves the existence of a one-to-one correspondence between the d-fold coverings of M branched over N and the equivalence classes of monodromies (i.e. transitive representations) $\omega : \pi_1(M - N) \to \Sigma_d$, where Σ_d denotes the symmetric group on d elements. Moreover, the Fox theorem gives the possibility of extending the concepts of regular or cyclic coverings to branched coverings. General references on the subject are [2], [18], [35] and [42].

The notion of branched covering can be extended to more general spaces (see [18] and [32]), including quasi-manifolds[1].

Branched coverings of spheres are of great interest, in particular as a method for representing manifolds. A classical result in this direction goes back to J.W.Alexander:

Proposition. [1] *Every n-manifold is a covering of $\mathbb{S}^n$ branched over the $(n-2)$-skeleton of a standard n-simplex.* □

Refinements of the Alexander theorem can be investigated in two different directions:

i) minimalize the number of sheets;

ii) find universal branching sets.

In dimension three, these approaches respectively lead to the possibility of representing all 3-manifolds by means of coloured knots and transitive permutation pairs.

In the first direction define a *coloured knot* as a pair (L, ω), where L is a knot and $\omega : \pi_1(\mathbb{S}^3 - L) \to \Sigma_3$ is a simple monodromy. In fact, (L, ω) can be drawn by colouring the arcs of a suitable diagram of L by $0, 1, 2$, so that the arc α is coloured c iff $\omega(\alpha)$ fixes c.

Representation theorem 1. [24], [33] *Every 3-manifold is a simple 3-fold covering of $\mathbb{S}^3$, branched over a knot. Thus, every 3-manifold is representable by coloured knots.* □

In the second direction, consider the handcuff-graph G embedded in $\mathbb{S}^3$ as in Figure 1.

FIGURE 1

Since $\pi_1(\mathbb{S}^3 - G)$ is a free group with two generators, the monodromy of any covering

[1] A quasi-manifold is a pseudo-manifold in which the star of every simplex is strongly connected ([19]).

$\tilde{M}$ of $\mathbb{S}^3$, branched over G, is given by a transitive permutation pair (σ, τ) of Σ_b, b being the degree of the covering. We will denote $\tilde{M}$ by $N_b(\sigma, \tau)$.

Representation theorem 2. [34] *Every 3-manifold is a covering of $\mathbb{S}^3$, branched over the graph G. Therefore, every 3-manifold is representable by transitive permutation pairs.* □

REMARK. It seems to be interesting to relate these two representation theories. In particular, the possibility of obtaining a coloured knot representing a 3-manifold M, starting from a pair (σ, τ) representing M, would give a combinatorial proof of Hilden-Montesinos theorem. In the opposite direction, the problem has been solved ([22]) by producing an algorithm for finding a transitive permutation pair (σ, τ) representing M, starting from of a coloured knot (L, ω) representing M. As a consequence, if L admits a "coloured diagram" with n crossings, then M is a $3n$-fold covering of $\mathbb{S}^3$, branched over G.

2. Edge-coloured graphs and gems

An $(n+1)$-*coloured graph* is a pair (Γ, γ), where:

- Γ is a finite connected regular multigraph of degree $n+1$;
- $\gamma : E(\Gamma) \to \Delta_n = \{0, 1, \dots, n\}$ is a proper edge-coloration (i.e. adjacent edges have different colours).

Every $(n+1)$-coloured graph represents a pseudosimplicial complex ([26]) $K(\Gamma)$ defined by:

- taking an n-simplex $\sigma(x)$ for each vertex $x \in V(\Gamma)$ and labelling its vertices by the elements of Δ_n;
- identifying, for every pair $x, y \in V(\Gamma)$ of c-adjacent vertices, the $(n-1)$-faces of $\sigma(x)$ and $\sigma(y)$ opposite to the vertices labelled by c.

The underlying space $|K(\Gamma)|$ is a quasi-manifold which is orientable iff Γ is bipartite ([17]). In dimension three, quasi-manifolds are also called *singular manifolds* ([34]).

If $B \subset \Delta_n$, $\#B = h \leq n$, there is a bijection between the components of the partial graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$ (called *h-residues*) and the $(n-h)$-simplices of $K(\Gamma)$ whose vertices are labelled by $\Delta_n - B$.

An *n-gem* is an $(n+1)$-coloured graph representing an n-manifold; every manifold M is representable by gems ([17], [41]).

Edge-coloured graph techniques provide a combinatorial way for representing manifolds; for general references see [3], [17], [29] and [47].

REMARK. An algorithm for obtaining, from a bipartite crystallization[2] Ω, a transitive permutation pair (σ, τ) such that $|K(\Omega)| \cong N(\sigma, \tau)$, is contained in [12] and [44]. The algorithm has been extended to the general case of bipartite 4-coloured graphs in [23].

A colour-preserving morphism (cp-morphism) between $(n+1)$-coloured graphs

$$f : (\Gamma, \gamma) \to (\Gamma', \gamma')$$

naturally induces a map

$$K(f) : |K(\Gamma)| \to |K(\Gamma')|$$

between the associated underlying spaces.

A cp-morphism $f : (\Gamma, \gamma) \to (\Gamma', \gamma')$ is said to be an *m-covering* $(1 \leq m \leq n)$ if the restriction of f to the m-residues is one-to-one ([45]). In particular:

- if $m = n$, then $K(f)$ is an ordinary covering,
- if $m = 1$, then $K(f)$ is a branched covering; moreover, if $|K(\Gamma')|$ is a manifold, the branching set is given by the $(n-2)$-simplices of $K(\Gamma')$ represented by the 2-residues (bicoloured cycles) of (Γ', γ') not ordinarily covered by f (i.e. the 2-residues whose counterimages via f have at least one component non-isomorphic to it).

3. Lins-Mandel graphs and spaces

The family of Lins-Mandel 4-coloured graphs

$$\mathfrak{G} = \{G(b, l, t, c) \mid b, l \in \mathbb{Z}^+,\ t \in \mathbb{Z}_{2l},\ c \in \mathbb{Z}_b\}$$

is defined in [29] by the following rules: the set of vertices of $G(b, l, t, c)$ is

$$V = \mathbb{Z}_b \times \mathbb{Z}_{2l}$$

and the coloured edges are obtained by these four fixed-point-free involutions on V:

[2] A crystallization is a gem with exactly $(n+1)$ n-residues ([17]).

$$\iota_0(i,j) = (i, j-(-1)^j),$$

$$\iota_1(i,j) = (i, j+(-1)^j),$$

$$\iota_2(i,j) = (i+\eta(j), 1-j),$$

$$\iota_3(i,j) = (i+c\eta(j-t), 1-j+2t);$$

where $\eta : \mathbb{Z}_{2l} \to \{-1,1\}$ is defined by $\eta(j) = \begin{cases} +1 & \text{if } 1 \leq j \leq l \\ -1 & \text{otherwise} \end{cases}$

For each $k \in \Delta_3$, we join the vertices $v, w \in V$ by a k-coloured edge iff $\iota_k(v) = w$.

Roughly speaking, the graph $G(b,l,t,c)$ is obtained by taking b copies C_i ($i \in \mathbb{Z}_b$) of the $\{0,1\}$-cycle of length $2l$ (so that $V(C_i) = \{(i,j) \mid j \in \mathbb{Z}_{2l}\}$) joined with:

- C_{i-1} and C_{i+1} by the 2-coloured edges,

- C_{i-c} and C_{i+c} by the 3-coloured edges.

FIGURE 2

Each $G(b,l,t,c) \in \mathfrak{G}$ is connected and bipartite; hence, it represents a (connected, orientable) singular 3-manifold $S(b,l,t,c)$. These spaces have been introduced as a combinatorial generalization of the lens spaces; in fact, $G(2,l,t,1)$ is the "normal" graph representing the lens space $L(l,t)$ ([15]). They have been intensively studied by M.R.Casali, A.Cavicchioli, A.Donati, L.Grasselli, D.L.Johnson-R.M.Thomas and, of course, S.Lins-A.Mandel (see [5], [6], [7], [8], [9], [10], [13], [21], [28] and [29]). We summarize the main results of these works:

1) if $(l,t) = 1$ and one of the following conditions holds:

— l is even and $c = \pm 1$,

— l is odd and $c = (-1)^t$,

then $S(b,l,t,c)$ is a 2-fold covering of $\mathbb{S}^3$, branched over a link;

2) $S(b,l,l-1,1) \cong S(b,l,1,-1)$ is the 2-fold covering of $\mathbb{S}^3$, branched over the torus link of type $\{b,l\}$, i.e. the Brieskorn manifold $M(b,l,2)$; if b and l are odd and coprime, these spaces are Seifert fibered homology spheres of Heegaard genus 2.

By making use of the symmetry of the Lins-Mandel graphs, it is easy to prove the existence of homeomorphisms between the associated spaces; as a consequence, we can restrict the range of some parameters, without loss of generality, as stated below.

(A) $S(b,l,t,c) \cong S(b,kl,kt,c) \implies \boxed{(l,t)=1}$

(B) $S(b,l,t,c) \cong S(b,l,t-l,-c) \implies \boxed{1 \le t \le l}$

(C) $S(b,l,t,c) \cong S(b,l,l-t,-c) \implies \boxed{t \text{ odd}}$

Thus, from now on, we restrict our attention to the subfamily

$$\widehat{\mathfrak{G}} = \{G(b,l,t,c) \in \mathfrak{G} \mid (l,t)=1,\ 1 \le t \le l,\ t \text{ odd}\}.$$

The cases (i) $c=0$ and (ii) $l=1$, $c=-1$ are "trivial", because the graphs admit planar regular embeddings ([20]), and therefore $S(b,l,t,0) \cong \mathbb{S}^3 \cong S(b,1,1,-1)$ (see [16]).

Since a singular 3-manifold is a 3-manifold iff its Euler characteristic vanishes ([43]), a (rather complicated) calculation gives a complete characterization of the Lins-Mandel graphs representing manifolds:

Proposition. [37] *A Lins-Mandel graph $G(b,l,t,c) \in \widehat{\mathfrak{G}}$ represents a 3-manifold iff either l is even or $c=0,-1$.* □

The main result concerning the topological properties of the Lins-Mandel spaces is the following:

Theorem. [38] *Let $G(b,l,t,c) \in \widehat{\mathfrak{G}}$ and $S(b,l,t,c)$ be the associated Lins-Mandel space.*

(a) $S(b,l,t,c)$ is a b-fold branched cyclic covering of $\mathbb{S}^3$.

(b) Suppose $c \neq 0$ (recall that the case $c=0$ is trivial). If $S(b,l,t,c)$ is a manifold, then the branching set is the 2-bridge link $\mathrm{b}(l,t)$; otherwise, the branching set is a θ-graph (Figure 3.a), non-trivially embedded when $l \neq 1$.

Sketch of proof. (a) The map $f_b : G(b,l,t,c) \to G(1,l,t,0)$, defined by $f_b(i,j)=(0,j)$, is the 1-covering induced by the action of the cyclic group $\mathbb{Z}_b$, generated by the cp-isomorphism $(i,j) \mapsto (i+1,j)$, on $G(b,l,t,c)$. Therefore, the associated map $K(f_b)$: $S(b,l,t,c) \to S(1,l,t,0) \cong \mathbb{S}^3$ is a branched cyclic covering map.

(b) When $S(b,l,t,c)$ is a manifold, it is easy to check that the set of the 2-residues of $G(1,l,t,0)$ not ordinarily covered by f_b does not depend on b and c; moreover it contains exactly four cycles. Incidence arguments on the lattice of the residues of $G(1,l,t,0)$ show that the 1-subcomplex R, whose edges are represented by these four cycles, is homeomorphic to $\mathbb{S}^1$ (resp. to $\mathbb{S}^1 \coprod \mathbb{S}^1$) iff l is odd (resp. is even). Since $G(1,l,t,0)$ and R only depend on l and t, R is the branching set of the 2-fold covering $K(f_2) : S(2,l,t,1) \cong L(l,t) \to \mathbb{S}^3$. The unicity of the representation of the lens spaces as 2-fold coverings of $\mathbb{S}^3$ ([27]) proves that the branching set is the two-bridge link $\mathbf{b}(l,t)$ (see [4]). For "non-manifolds", the branching set is represented by five 2-residues and is homeomorphic to a θ-graph; again, the embedding only depends on l and t (the description of the embedding can be found in [39]). □

FIGURE 3

As a consequence of this theorem and of the Smith conjecture ([36]), we can completely characterize the spheres among Lins-Mandel spaces:

Corollary. [38] *A Lins-Mandel manifold $S(b,l,t,c)$ is homeomorphic to $\mathbb{S}^3$ iff either $c=0$ or $l=1$.* □

REMARK. The monodromy $\omega : \pi_1(\mathbb{S}^3 - \mathbf{b}(l,t)) \to \Sigma_b$, associated to the covering, is defined by

$$\omega(m_1)(i) = i+1,$$

$$\omega(m_2)(i) = i-c;$$

where m_1 and m_2 are meridians associated to the two bridges of $\mathbf{b}(l,t)$ (see Figure 4). Note that, if l is odd, $\mathbf{b}(l,t)$ is a knot and therefore m_1 and m_2 are associated to the same component of the branching set; this explains why l odd implies $c=-1$.

FIGURE 4

Let now $M(L,\omega)$ denote the b-fold cyclic covering of $\mathbb{S}^3$, branched over an oriented link L, and defined by the monodromy $\omega : \pi_1(\mathbb{S}^3 - L) \to \Sigma_b$. The following notions are given in [30]:

(a) $M(L,\omega)$ is *strictly-cyclic* if $\omega(m_i) = \omega(m_j)$, for every meridian pair m_i, m_j;

(b) $M(L,\omega)$ is *almost-stricly-cyclic* if $\omega(m_i) = \omega(m_j)^{\pm 1}$, for every meridian pair m_i, m_j;

(c) $M(L,\omega)$ is *meridian-cyclic* if $\text{ord}(\omega(m_i)) = b$, for every meridian m_i;

(d) $M(L,\omega)$ is *singly-cyclic* if there exists a meridian m_i such that $\text{ord}(\omega(m_i)) = b$.

It is straightforward that:

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \text{(d)} \Rightarrow \text{cyclic;}$$

moreover, the five notions are equivalent when L is a knot.

As a direct consequence of the above theorem and remark, we obtain:

Corollary. [38] *The Lins-Mandel manifold $S(b,l,t,c)$ is a branched singly-cyclic covering of $\mathbb{S}^3$. In particular:*

- *if $c = -1$, the covering is strictly-cyclic;*
- *if $c = \pm 1$, the covering is almost-strictly-cyclic;*
- *if $(b,c) = 1$, the covering is meridian-cyclic.* □

REMARKS. (1) The manifolds $S(b,l,t,c)$ with $(b,c) = 1$ are precisely the Lins-Mandel manifolds whose corresponding gems $G(b,l,t,c)$ are crystallizations ([5]).

(2) The Minkus manifolds $M_n(k,h)$, investigated in [31], are particular cases of Lins-Mandel manifolds. Actually, we have $M_n(k,h) \cong S(n,k,h,-1)$.

To end this section, we present necessary and sufficient conditions for the isomorphism between Lins-Mandel gems. As a consequence, we get sufficient conditions for the homeomorphism between Lins-Mandel manifolds (different from the sphere and lens spaces).

Proposition. *Let $G = G(b,l,t,c)$, $G' = G(b',l',t',c') \in \widehat{\mathfrak{G}}$ be gems, with $l,l' > 2$. Then G is isomorphic to G' iff either*

$$b' = b,\ l' = l,\ t' = \pm t^{\pm 1},\ c' = c^s$$

or

$$b' = b,\ l' = l,\ t' = \pm t^{\pm 1} + l,\ c' = -c^s;$$

where

$$s = \begin{cases} +1 & \text{if } (b,c) \neq 1 \\ \pm 1 & \text{if } (b,c) = 1 \end{cases}.$$

Hence, if one of the above conditions holds, the manifolds $S(b,l,t,c)$ and $S(b',l',t',c')$ are homeomorphic. □

4. Generalized Lins-Mandel spaces

The Lins-Mandel family only contains singly-cyclic coverings. This makes natural the attempt of extending it, in order to obtain the whole class of cyclic coverings of $\mathbb{S}^3$, branched over two-bridge links.

We define a new class of 4-coloured graphs depending on five parameters:

$$\tilde{\mathfrak{G}} = \{\tilde{G}(b,l,t,c,c') \mid b,l \in \mathbb{Z}^+,\ t \in \mathbb{Z}_{2l},\ c,c' \in \mathbb{Z}_b,\ \gcd(b,c,c') = 1\}.$$

Each $\tilde{G}(b,l,t,c,c')$ is defined by the following four fixed-point-free involutions on the set $V = \mathbb{Z}_b \times \mathbb{Z}_{2l}$:

$$\tilde{\iota}_0 = \iota_0,$$

$$\tilde{\iota}_1 = \iota_1,$$

$$\tilde{\iota}_2(i,j) = \big(i + c'\eta(j), 1-j\big),$$

$$\tilde{\iota}_3 = \iota_3;$$

it represents a (connected, orientable) singular 3-manifold $\tilde{S}(b,l,t,c,c')$.

FIGURE 5

We easily obtain the following results:

1) $\tilde{S}(b,l,t,c,1) \cong S(b,l,t,c)$;
2) $\tilde{S}(b,l,t,0,c') \cong \tilde{S}(b,l,t,c,0) \cong \tilde{S}(b,1,1,-c',c') \cong \mathbb{S}^3$;
3) $\tilde{S}(b,l,t,c,c') \cong \tilde{S}(b,l,t,c',c)$;
4) If $(b,c) = 1$ or $(b,c') = 1$, then there exists $c'' \in \mathbb{Z}_b$ such that $\tilde{S}(b,l,t,c,c') \cong S(b,l,t,c'')$; thus, the generalization is effective for the cases $(b,c) \neq 1 \neq (b,c')$;
5) As in Section 3, we can assume, without loss of generality, $(l,t) = 1$, $1 \leq t \leq l$ and t odd.

The characterization of the manifolds among Lins-Mandel generalized spaces is similar to the previous one:

Proposition. [38] *The graph $\tilde{G}(b,l,t,c,c')$ represents a 3-manifold iff either l is even or at least one of the following conditions holds: (i) $c=0$, (ii) $c'=0$, (iii) $c=-c'$.* $\square$

Theorem. [38] *The 3-manifold $\tilde{S}(b,l,t,c,c')$, with $c \neq 0 \neq c'$, is the b-fold cyclic covering of $\mathbb{S}^3$, branched over $\mathbf{b}(l,t)$. The associated monodromy is defined by:*

$$\omega(m_1)(i) = i + c',$$

$$\omega(m_2)(i) = i - c.$$

Therefore, the class of generalized Lins-Mandel manifolds $\tilde{S}(b,l,t,c,c')$, with $c \neq 0 \neq c'$, is precisely the class of all cyclic coverings of $\mathbb{S}^3$ branched over the two-bridge links (with the exception of the trivial link with two components). $\square$

Corollary. [38] *The 3-manifold $\tilde{S}(b,l,t,c,c')$ is a sphere iff either $c=0$ or $c'=0$ or $l=1$.* $\square$

5. Further generalizations

The attempt of obtaining a class of gems representing all coverings of $\mathbb{S}^3$, branched over two-bridge links, leads to the following extension.

Take $b,l \in \mathbb{Z}^+$, $t \in \mathbb{Z}_{2l}$, with $(l,t)=1$. Let (σ,τ) be a transitive permutation pair of Σ_b. Define the following four fixed-point-free involutions on $V = \mathbb{Z}_b \times \mathbb{Z}_{2l}$:

$$\bar{\iota}_0 = \iota_0,$$

$$\bar{\iota}_1 = \iota_1,$$

$$\bar{\iota}_2(i,j) = \left(\sigma^{\eta(j)}(i), 1-j\right),$$

$$\bar{\iota}_3(i,j) = \left(\tau^{-\eta(j-t)}(i), 1-j+2t\right).$$

Denote by $\bar{G}(b,l,t,\sigma,\tau)$ the resulting 4-coloured graph and by $\bar{S}(b,l,t,\sigma,\tau)$ the associated space.

FIGURE 6

Lemma. [39] *a) If v is the cyclic permutation $(0\ 1\ 2\ \cdots\ b-1)$, then $\bar{S}(b,l,t,v^{c'},v^{-c}) \cong \bar{S}(b,l,t,c,c')$;*

b) If $\sigma = 1$ or $\tau = 1$, then $\bar{S}(b,l,t,\sigma,\tau) \cong \mathbb{S}^3$;

c) $\bar{S}(b,l,t,\sigma^{-1},\tau^{-1}) \cong \bar{S}(b,l,t,\sigma,\tau) \cong \bar{S}(b,l,t,\tau,\sigma)$. □

Let φ be the permutation $\varphi = \sigma^{\eta(t)}\tau^{\eta(2t)}\sigma^{\eta(3t)}\tau^{\eta(4t)}\cdots\sigma^{\eta\left((2l-1)t\right)}\tau^{\eta(2lt)}$.

Theorem. [39] *$\bar{S}(b,l,t,\sigma,\tau)$ is the b-fold covering of $\mathbb{S}^3$ branched over a 1-subcomplex R with the following description:*

1) if l is odd and

a) $\sigma,\tau,\varphi \neq 1$, then R is a θ-graph (Figure 3.a), non-trivially embedded when $l \neq 1$;

b) $\sigma,\tau \neq 1$ and $\varphi = 1$, then R is the two-bridge knot $\mathbf{b}(l,t)$;

c) $\sigma = 1$ or $\tau = 1$, then R is the trivial knot;

2) if l is even and

a) $\sigma,\tau,\varphi \neq 1$, then R is a handcuff-graph (Figure 3.b), non-trivially embedded;

b) $\sigma,\tau \neq 1$ and $\varphi = 1$, then R is the two-bridge link $\mathbf{b}(l,t)$;

c) $\sigma = 1$ or $\tau = 1$, then R is the trivial knot.

In any case, the fundamental group $\pi_1(\mathbb{S}^3 - R)$ admits a presentation with two generators X, Y and, in the cases 1.a and 2.a, it is the free group $< X, Y\,;\, \emptyset >$. The monodromy associated to the covering is defined by

$$\omega(X) = \sigma,$$
$$\omega(Y) = \tau. \qquad \square$$

REMARKS. (a) If $l = 2$ and $t = 1$, the branching set R is precisely the universal graph G of Montesinos (Figure 1). Moreover, ω is the monodromy of the branched covering $N_b(\sigma,\tau)$; so we have the homeomorphism $\bar{S}(b,2,1,\sigma,\tau) \cong N_b(\sigma,\tau)$. Since each (singular) 3-manifold is homeomorphic to a suitable $N_b(\sigma,\tau)$, the subclass of graphs $\{\bar{G}(b,2,1,\sigma,\tau)\}$ is a (very symmetric) "universal" class of 4-coloured graphs representing all singular 3-manifolds.

(b) The class $\bar{\mathfrak{G}}_{l,t} = \{\bar{G}(b,l,t,\sigma,\tau) \mid \sigma,\tau \neq 1,\ \varphi = 1,\ b \in \mathbb{Z}^+\}$ precisely represents all coverings of $\mathbb{S}^3$, branched over the two-bridge link $\mathbf{b}(l,t)$. Thus, every space of $\bar{\mathfrak{G}}_{l,t}$ is a manifold. Moreover, since a non-toroidal two-bridge link is universal ([25]), every subclass $\{\bar{\mathfrak{G}}_{l,t} \mid t \neq \pm 1 \pmod{l}\}$ is a "universal" class of gems representing all 3-manifolds.

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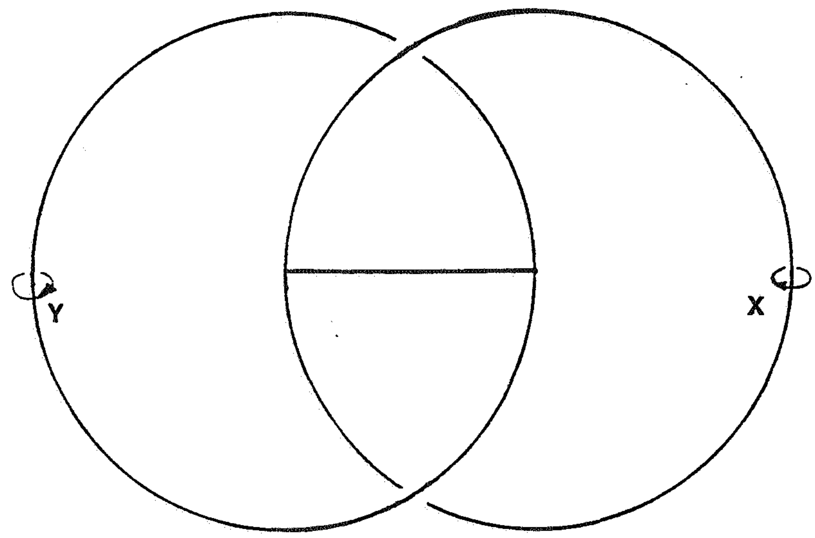


Figure 1. The Montesinos universal graph G

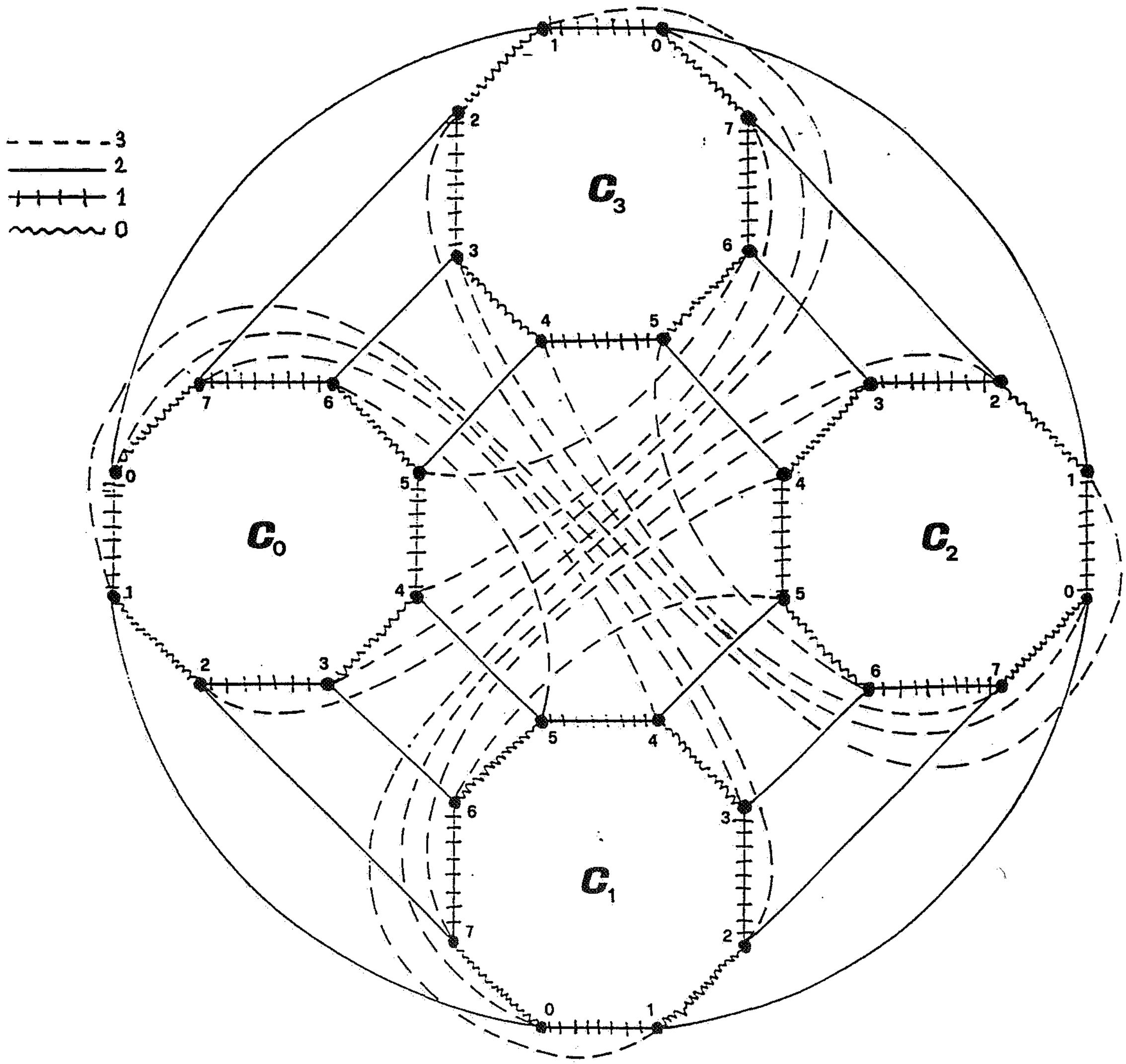
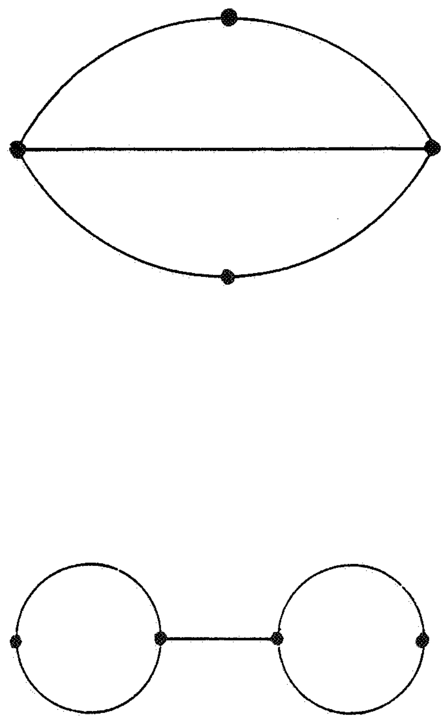


Figure 2. The Lins-Mandel gem $G(4,4,1,3)$



(a)

(b)

Figure 3. (a) A θ-graph (trivially embedded) — (b) A handcuff-graph (trivially embedded)

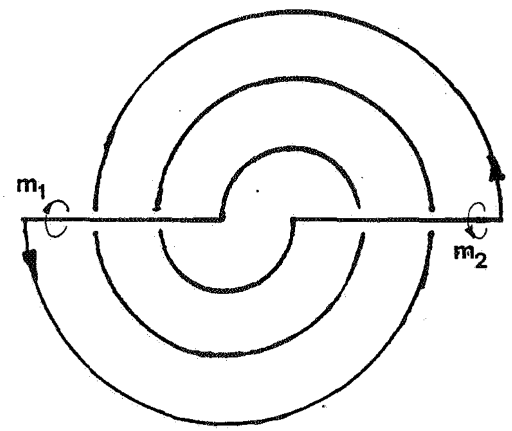


Figure 4. The two-bridge knot $\mathbf{b}(3,1)$

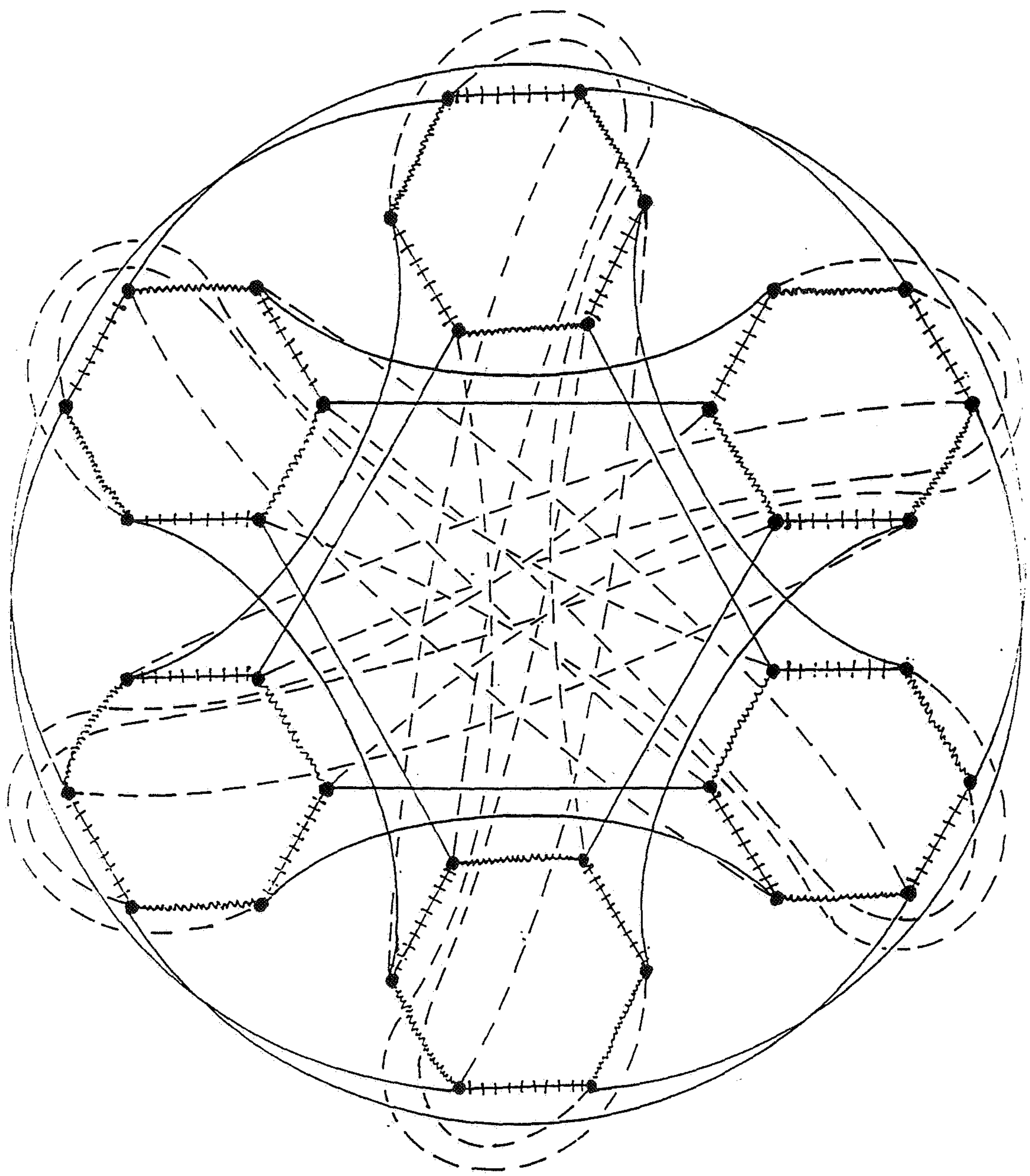


Figure 5. The graph $\tilde{G}(6,3,2,3,2)$

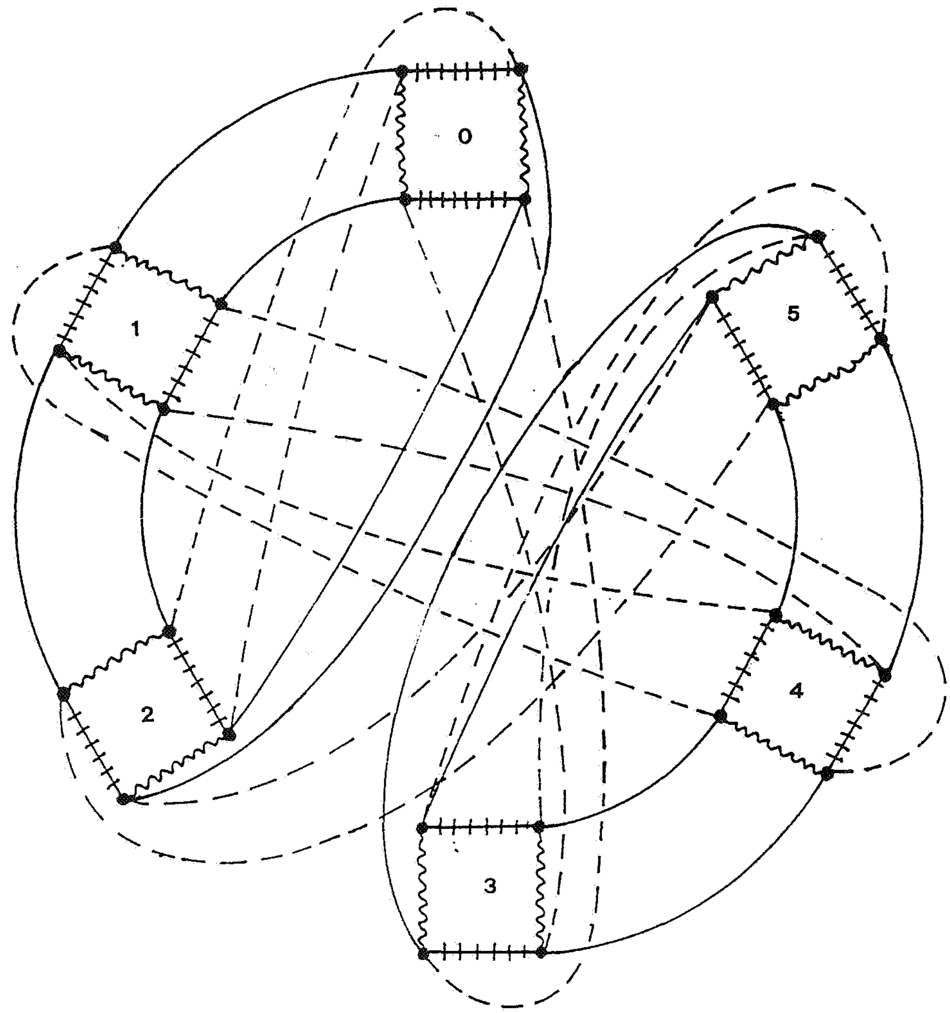


Figure 6. $\bar{G}(6, 2, 1, (0\ 1\ 2)(3\ 4\ 5).(0\ 2\ 5\ 3)(1\ 4))$— represents $S^2 \times S^1$ —