UNIVERSIDAD NACIONAL DE EDUCACIÓN A DISTANCIA

DISERTACIONES DEL SEMINARIO DE MATEMÁTICAS FUNDAMENTALES

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CRYSTALLIZATIONS AND OTHER MANIFOLD REPRESENTATIONS

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Crystallizations and other manifold representations

by

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1. INTRODUCTION.

The aim of this survey paper is to describe a method for representing pl-manifolds by means of edge-coloured graphs - called "crystallizations" - and to relate this theory with other well-known representations defined in dimension three (Heegaard diagrams and branched coverings of the 3-sphere S^3). Manifolds will be assumed closed and connected; in ch.5 (for seek of conciseness) and 6 we also suppose they are orientable.

We point out from the beginning that fundamental features of crystallization theory are:

- its validity in every dimension: this leads to the possibility of extending 3-manifold invariants to dimension n (ch.5);
- its combinatorial nature which makes it particularly oriented to the enumeration of special classes of manifolds (**[LM]**, **[CG2]**, **[C2]**);
- the possibility of easily reading the classical topological invariants of a manifold directly from a representing graph (ch.3);
- the existence of an "equivalence criterion" between graphs which translates the homeomorphism of the represented manifolds (ch.4).

2. EDGE-COLOURED GRAPHS AND ASSOCIATED COMPLEXES.

An $(n+1)$-*coloured graph* is a pair (Γ,γ), where Γ is a regular graph of degree $n+1$ and $\gamma: E(\Gamma) \longrightarrow \Delta_n = \{0,1,...,n\}$ is an edge-coloration (such that adjacent edges have not the same colour).

An n-dimensional ball-complex $K(\Gamma)$ triangulating an n-pseudo-manifold can be associated to a given (n+1)-coloured graph (Γ,γ) by means of the following rules:

(1) take an n-simplex $\sigma(x)$ for each vertex $x\in V(\Gamma)$ and label its vertices by Δ_n;

(2) if $x,y\in V(\Gamma)$ are joined by a c-coloured edge, identify the (n-1)-faces of $\sigma(x)$ and $\sigma(y)$ opposite to the vertices labelled by c, so that equally labelled vertices are identified together.

Even if the balls of $K(\Gamma)$ are simplexes, the resulting complex $K(\Gamma)$ is not in general a simplicial one since the intersection of two simplexes may be the union of more than one maximal face; thus, it is simply a *pseudocomplex*, in the sense of **[HW,pag.49]**. We say that the graph (Γ,γ) *represents* $K(\Gamma), |K(\Gamma)|$ and every homeomorphic space.

Remarks.

- By dropping the regularity condition for Γ, we may represent pseudomanifolds with (non-empty) boundary; in fact, if x is a vertex with no adjacent c-coloured edges, then the (n-1)-face of $\sigma(x)$ opposite to the c-labelled vertex of $K(\Gamma)$ is a boundary face.

- Set $\hat{c} = \Delta_n \setminus \{c\}$. If $B \subseteq \Delta_n$, Γ_B denotes the subgraph of (Γ,γ) defined by $V(\Gamma_B)=V(\Gamma)$, $E(\Gamma_B)=\gamma^{-1}(B)$. The components of $\Gamma_{\{i,j\}}$ are bicoloured cycles, for each $i,j\in\Delta_n$.

If the cardinality $\#B$ of B is $h\leq n$, then there is a bijection δ between the set of components of Γ_B and the set of (n-h)-simplexes of $K(\Gamma)$ whose vertices are labelled by $\Delta_n \setminus B$; moreover, δ reverses inclusion. This remark states, in particular, that Γ is isomorphic with the 1-skeleton of the dual complex of $K(\Gamma)$ and that the number of c-labelled vertices of $K(\Gamma)$ is equal to the number of components of $\Gamma_{\hat{c}}$, for each $c \in \Delta_n$.

- The construction of $K(\Gamma)$ gives a coloration on the vertex set $S_0(K)$ of $K(\Gamma)$ by means of n+1 colours (i.e. a map $\xi : S_0(K) \longrightarrow \Delta_n$ which is injective on each simplex of $K(\Gamma)$. Given an n-pseudomanifold M, triangulated by a pseudocomplex K with such a coloration on its vertex set $S_0(K)$, the construction can be easily reversed yielding an (n+1)-coloured graph (Γ,γ) such that $K(\Gamma)=K$.

Existence theorems.

Proposition 1. *Every* n*-pseudomanifold* M *may be represented by an* (n+1)*-coloured graph.*

In fact, if K is a simplicial triangulation of M and K′ is its barycentric subdivision, it is easy to obtain a coloration ξ of the vertices of K′ by labelling each vertex v of K′ by the dimension of the simplex of K whose barycenter is v. The inverse construction applied to (K',ξ) yields a graph (Γ,γ) such that $|K(\Gamma)|=|K'|=M$.

A basic idea by Pezzana is to minimize the number of vertices in $K(\Gamma)$, that is to consider n-pseudocomplexes with exactly n+1 vertices (the so called *contracted* complexes). Since, as a consequence of the above remarks, $K(\Gamma)$ is contracted if and only if each $\Gamma_{\hat{c}}$ is connected, it seems natural to say that a graph (Γ,γ) satisfying this last condition is contracted too. A *crystallization* of an n-manifold M is any contracted (n+1)-coloured graph representing M.

Proposition 2. *Every* n*-manifold* M *is representable by crystallizations.*

The original proof was given by Pezzana **[P1]** by an explicit construction of a contracted triangulation of M, starting from a simplicial one. Pezzana's result makes "crystallization theory" a general representation for pl-manifolds; a survey on this theory is exposed in **[FGG]**.

3. CHARACTERIZATIONS AND TOPOLOGICAL INVARIANTS.

A fundamental problem is the possibility of characterizing the edge-coloured graphs representing manifolds. A general result is the following one.

Proposition 3. *An* $(n+1)$*-coloured graph* (Γ,γ) *represents an* n*-manifold if and only if each component* R *of each* $\Gamma_{\hat{c}}$ $(c\in\Delta_n)$ *represents* S^{n-1}.

This is because if v is the vertex of $K(\Gamma)$ corresponding to R via the bijection δ, R represents, as an n-coloured graph, the link of v in $|K(\Gamma)|$.

Proposition 3 does actually provide a *recognition algorithm* for n-manifold graphs only if $n=2,3$, since we do not know any algorithm for characterizing graphs representing S^{n-1} if $n>3$.

For $n=2$, every 3-coloured graph represents a 2-manifold.

Since, in general, the bijection δ allows us to compute the Euler characteristic of $K(\Gamma)$, if $n=3$ the most direct algorithm to check if (Γ,γ) represents a 3-manifold is to test if $\chi(K(\Gamma))=0$. Quick methods can be easily obtained for recognizing crystallizations among contracted 4-coloured graphs **[G1]**.

Orientability.

Proposition 4. [CGP] *Let* (Γ,γ) *be any* $(n+1)$*-coloured graph representing an* n*-pseudomanifold* M. *Then* M *is orientable if and only if* Γ *is bipartite.*

Fundamental group.

Two different methods are known for computing the fundamental group $\pi_1(M)$ of an n-manifold M directly from an (n+1)-coloured graph representing M.

(1) **[G2]** Let (Γ,γ) be a crystallization of M. Choose two colours $i,j\in\Delta_n$ and call $X=\{x_1,\ldots,x_q\}$ the set of all components of $\Gamma_{\Delta_n\setminus\{i,j\}}$ but one arbitrarily chosen. If n=2, let y_1 be the only cycle of $\Gamma_{\{i,j\}}$. If n>2, call $\{y_1,\ldots,y_m\}$ the set of all components of $\Gamma_{\{i,j\}}$ but one arbitrarily chosen; fix a running direction and a starting point for each of them. Compose the word r_h on X from the cycle y_h by the following rule: follow the chosen direction starting from the chosen vertex and write consecutively every generator you meet, with exponent +1 or -1 according to i or j being the colour of the edge by which you run into the generator. $P=\langle X \mid r_1,\ldots,r_m\rangle$ is a presentation of $\pi_1(M)$. Since, if n=3, the number of components of $\Gamma_{\{i,j\}}$ is equal to the number of components of $\Gamma_{\Delta_n\setminus\{i,j\}}$, then m=q and hence P has non negative deficiency.

(2) **[Gr]** If (Γ,γ) is an (n+1)-coloured graph representing M and $\Gamma_{\hat{c}}$ is connected, for some $c\in\Delta_n$, a presentation $P'=\langle X'\mid R'\rangle$ of $\pi_1(M)$, called *c-edge presentation*, can be obtained in the following way:

- (*) the generators of X′ are the c-coloured edges, arbitrarily oriented;
- (**) the relators of R′ are obtained by walking along the components of $\Gamma_{\{i,c\}}$, for each $i\in\Delta_n\setminus\{c\}$, giving the exponent +1 or -1 to each generator whether the orientation of the component is coherent or not with the orientation of the generator.

Since we have, in general, $\#X < \#X'$, the first method is useful in practical computation: nevertheless, the second one plays an important role in the theory.

4. THE EQUIVALENCE CRITERION : FERRI-GAGLIARDI MOVES.

Given two (n+1)-coloured graphs (Γ,γ), (Γ',γ'), an isomorphism $\psi:\Gamma\longrightarrow\Gamma'$

is called a *colour-isomorphism* if there is a bijection $\phi:\Delta_n \longrightarrow \Delta_n$ such that $\gamma'\circ\psi=\phi\circ\gamma$. In this case (Γ,γ) and (Γ',γ') are said to be *isomorphic*.

Colour-isomorphism of edge-colored graphs implies homeomorphism of the represented pseudomanifolds, but there are many contracted triangulations, leading to non-isomorphic crystallizations of the same manifold.

The problem of finding an equivalence criterion, internal to the theory, which translates the notion of homeomorphism type has been solved **[FG1]** by giving a set of two moves such that, for any two crystallizations of the same manifold, a finite sequence of such moves exists which takes one crystallization to the other.

Applications.

Even if these moves give no algorithm to recognize if two given crystallizations represent the same manifold, they allow the search of:

- "normal forms" for crystallizations, such that an existence theorem for all manifolds still holds **[BDG]**;
- new topological invariants directly computed from the crystallization (to check if an arbitrary structure associated to the crystallizations is a topological invariant it suffices to test if it is invariant under moves).

The moves.

Given an $(n+1)$-coloured graph (Γ,γ), a subgraph θ of Γ formed by two vertices X,Y joined by h edges $(1\leq h\leq n)$ with colours $c_1,\ldots,c_h$ is a *dipole of type* h if X and Y belong to distinct components of $\Gamma_{\Delta_n\setminus\{c_1,\ldots,c_h\}}$. If h=1 or h=n, the dipole is degenerate.

Cancelling θ means:

(a) deleting vertices and edges of θ;

(b) welding the "hanging" edges of the same colour.

Adding θ means the inverse process.

In a crystallization:

move I is the addition or cancellation of a non degenerate dipole;

move II is the addition of a (degenerate) dipole of type 1 (yielding a non-contracted graph) followed by the cancellation of a different dipole of type 1 involving the same colour.

Proposition 5. *If* M,M′ *are* n-*manifolds and* (Γ,γ), (Γ',γ') *are two crystallizations of them, then* M *is homeomorphic with* M′ *if and only if* (Γ,γ) *and* (Γ',γ') *can be joined by a finite sequence of moves I and/or II.*

5. RELATIONS WITH HEEGAARD REPRESENTATION THEORY.

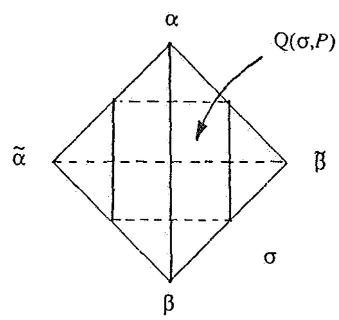
This 3-manifold representation theory is based on the following result:

Proposition 6. [H] *Every* 3-*manifold* M *is the identification space* $Y_g \cup_\phi \tilde{Y}_g$ *obtained by glueing the boundaries of two handlebodies* Y_g, $\tilde{Y}_g$ *of genus* g, *via a suitable homeomorphism* $\phi:\partial Y_g \longrightarrow \partial \tilde{Y}_g$.

The least integer g such that $M=Y_g \cup_\phi \tilde{Y}_g$ is a topological invariant - the *Heegaard genus* - of the 3-manifold M.

Since isotopic homeomorphisms give the same manifold and since each homeomorphism ϕ_g is completely determined by the g images of a complete system of meridians on Y_g, the result allows to represent M by a triad (F_g,x,y), where x and y are complete system of meridians - the *Heegaard diagram* of M - drawn on the surface $F_g=\partial Y_g=\partial \tilde{Y}_g$.

An easy proof of proposition 6 can be obtained by considering the pseudocomplex $K(\Gamma)$ associated to a 4-coloured graph (Γ,γ) representing M **[P2]**. In fact, if *P* is one of the three partitions $\{\{\alpha,\beta\},\{\tilde{\alpha},\tilde{\beta}\}\}$ of Δ_3, then we can decompose each tetrahedron σ of $K(\Gamma)$ in two prisms as indicated in the following figure:



If $Q(\sigma,P)$ is the 4-gon which is the common face of the two prisms decomposing σ, then

$$\bigcup_{\sigma\in S_3(K(\Gamma))} Q(\sigma,P)$$

is a closed surface F, depending on P, which splits M in two handlebodies whose common boundary is F.

This proof gives a straight relation between edge-coloured graphs representing M and Heegaard splittings of M. Moreover, if (Γ,γ) is a crystallization of M, the choice of a cyclic permutation $\varepsilon=(\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3)$ on Δ_3 directly produces Heegaard diagrams of M. In fact, given ε (or its inverse), let P_ε be the partition $\left\{\{\varepsilon_0,\varepsilon_2\}\{\varepsilon_1,\varepsilon_3\}\right\}$ of Δ_3 and consider the Heegaard surface

$$F_\varepsilon = \bigcup_{\sigma\in S_3(K(\Gamma))} Q(\sigma,P_\varepsilon)$$

of M. Since Γ is the 1-skeleton of the dual complex of $K(\Gamma)$, Γ may be cellularly imbedded in F_ε [W], [S] so that each region is bounded by a cycle of edges of Γ, alternatively coloured by a pair $(\varepsilon_i,\varepsilon_{i+1})$ of consecutive colours $(i\in Z_4)$. Further, the images of the (0,2)-coloured (resp. (1,3)-coloured) cycles of (Γ,γ) but one arbitrarily chosen gives a complete system x (resp. y) of meridians on F_ε. Thus, the triad (F.x,y) is a Heegaard diagram of M.

The inverse construction, leading to a crystallization from a given Heegaard diagram, is described in [G3].

The above arguments suggest a way for extending the concept of genus to dimension n.

For, define a *regular imbedding* of an (n+1)-coloured graph (Γ,γ) into a closed surface F as a cellular imbedding of Γ in F such that, for a given cyclic permutation $\varepsilon=(\varepsilon_0,...,\varepsilon_n)$ of Δ_n, each region is bounded by the image of a cycle whose edges are alternatively coloured by $(\varepsilon_i,\varepsilon_{i+1})$. The *regular genus* $\rho(\Gamma,\gamma)$ of a crystallization (Γ,γ) of an n-manifold M^n is defined as the minimal genus of a surface in which (Γ,γ) regularly imbeds and the regular genus $\mathcal{G}(M^n)$ of M^n is defined by:

$$\mathcal{G}(M^n) = \min \left\{\rho(\Gamma,\gamma) \mid (\Gamma,\gamma) \text{ crystallization of } M^n\right\}.$$

If g(F) (resp. $\mathcal{H}(M)$) denotes the genus (resp. the Heegaard genus) of the surface F (resp. of the 3-manifold M) we have:

Proposition 7. **[G3]** *For every surface* F, $\mathcal{G}(F)=g(F)$;

for every 3-manifold M, $\mathcal{G}(M)=\mathcal{H}(M)$.

Thus, the regular genus may be considered as an *extension* of the concept of genus to dimension n.

Some remarkable results about the regular genus are the followings ones.

Proposition 8. $\mathcal{G}(M^n)\geq rk(M^n)$ **[BM]**;

$\mathcal{G}(M^n) = 0 \Longleftrightarrow M^n = S^n$ **[FG2]**;

$\mathcal{G}(M^4) = 1 \Longleftrightarrow M^4 = S^3\times S^1$ **[Ca]**.

6. RELATIONS WITH BRANCHED COVERING THEORY.

It is well known that every 3-manifold is a covering of the 3-sphere S^3 branched over a link **[A]** **[Hi]** **[M]**. Hence, every 3-manifold may be represented by a monodromy map $\omega:\pi_1(S^3-L) \longrightarrow S_d$, L being a suitable link in S^3 and S_d being the symmetric group on d elements.

A construction is known which allows to obtain, starting from a bridge-presentation of a link L and a monodromy map $\omega:\pi_1(S^3\text{-}L) \longrightarrow S_d$, a 4-coloured graph representing the 3-manifold $M(L,\omega)$ determined by (L,ω) [CG1].

For this aim, it is useful to define (a,b,c)-graph C' of length 4h a 3-coloured graph obtained in the following way:

- let C be a cycle with 4h vertices whose edges are alternatively coloured by the colours a and b;

- if $\{\sigma,\tau\}$ is a pair of opposite a-coloured edges of C, let α be the involutory automorphism on V(C) induced by the symmetry whose axis is the straight line passing through the barycenters of σ and τ. Then join each vertex $v\in V(C)$ with $\alpha(v)$ by a c-coloured edge. C is said to be *the boundary of* C'.

Let $P(L)=(b_1^+,\dots,b_m^+;\ b_1^-,\dots,b_m^-)$ be a connected planar projection of an m-bridge-presentation of a link L such that the projection b_i^+ of the bridges are contained in the same straight line r of the plane Π containing P(L). For each $i\in\{1,2,\dots,m\}$, let h_i^+ (resp. h_i^-) be the number of crossings of b_i^+ (resp. of the arc b_i^-). Set $\sum_{i=1}^{m} h_i^+ = \sum_{i=1}^{m} h_i^- = n$. Let $<X=\{x_1,\dots,x_m\}|R>$ be the presentation of $\pi_1(S^3\setminus L)$ whose generators x_i biunivocally correspond to the projections b_i^+ of the bridges of L.

First step.

Let (Γ,γ) be the 4-coloured graph obtained in the following way:

- draw on Π a (2,0,3)-graph L_i' of length $4(h_i^+ +1)$ for each b_i^+ of P(L), so that the end-points of b_i^+ are respectively contained in the interior of the two cells bordered by the (2,3)-bicoloured cycles of L_i' of length two and b_i^+ is contained in the cell bordered by the boundary L_i of L_i';

- draw the 1-coloured edges so that each arc b_i^- of P(L) is contained in the interior of the cell bordered by a (1,3)-bicoloured component T_i of length $4(h_i^- +1)$ (the boundary of a (3,1,2)-graph T_i').

Let H be the 1-dimensional subcomplex of $K(\Gamma)$ whose edges are represented by the 2m components (two for each b_i^+) of $\Gamma_{\{2,3\}}$ of length two. The pair $(K(\Gamma),H)$ triangulates (S^3,L).

Second step.

Orient every 3-coloured edge of (Γ,γ) so that its first (resp. second) vertex belongs to the lower (resp. upper) half-plane of Π determined by the line r containing all b_i^+'s.

The 4-coloured graph (Γ',γ') representing $M(L,\omega)$ is obtained in the following way:

- $V(\Gamma') = V(\Gamma) \times \{1,2,....,d\}$;

- for each $c\in\Delta_2$, the vertices $(v,i),(w,j)$ are c-adjacent in (Γ',γ') if and only if v,w are c-adjacent in (Γ,γ) and i=j;

- the vertices $(v,i),(w,j)$ are 3-adjacent in (Γ',γ') if and only if v,w are respectively the first and the second vertex of a 3-coloured edge σ of (Γ,γ) intersecting b_h^+ and $\omega(x_h)(i)=j$.

The map $f : V(\Gamma') \longrightarrow V(\Gamma)$ defined by $f(v,i) = v$ induces a pl-map $K(f) : |K(\Gamma')| \longrightarrow |K(\Gamma)|$ which is the covering projection of $M(L,\omega)=|K(\Gamma')|$ on $S^3= |K(\Gamma)|$ branched over L.

An explicit presentation of the fundamental group of *M(L,ω)*, depending on the presentation $<X|R>$ of $\pi_1(S^3\setminus L)$ and on the permutations $\omega(x_i)$, $x_i\in X$, is obtained in **[C1]** by making use of the above construction.

Note that the second step of the construction can be easily *extended to dimension n* and applied to an arbitrary pair $(K(\Gamma),H)$, (Γ,γ) being an (n+1)-coloured graph representing the n-manifold $|K(\Gamma)|$, with $\Gamma_{\hat{n}}$ connected, and H being the (n-2)-dimensional subcomplex of $K(\Gamma)$ represented by a given set of bicoloured components of $\Gamma_{\{n-1,n\}}$. The n-manifold $|K(\Gamma')|$ represented by the resulting graph (Γ',γ') is a covering of $|K(\Gamma)|$ branched over H.

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These notes collect some of the talks given in the Seminario del Departamento de Matemáticas Fundamentales de la U.N.E.D. in Madrid. Up to now the following titles have appeared:

1 **Luigi Grasselli,** Crystallizations and other manifold representations.
2 **Ricardo Piergallini**, Manifolds as branched covers of spheres.
3 **Gareth Jones**, Enumerating regular maps and hypermaps.
4 **J.C.Ferrando, M.López-Pellicer**, Barrelled spaces of class N and of class χ_0
5 **Pedro Morales**, Nuevos resultados en Teoria de la medida no conmutativa.
6 **Tomasz Natkaniec**, Algebraic structures generated by some families of real functions.
7 **Gonzalo Riera**, Algebras of Riemann matrices and the problem of units.
8 **Lynne D. James**, Representations of Maps.
9 **Grzegorz Gromadzki**, On supersoluble groups acting on Klein surfaces.
10 **Maria Teresa Lozano**, Flujos en 3-variedades.