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Fuglede's theorem in variable exponent Sobolev space

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Abstract

Consider an open set $\Omega \subset \mathbb{R}^n$ and a function in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$. We show that there exists a family of curves Γ with zero $p(\cdot)$-modulus such that the quasi-continuous representative of u is absolutely continuous on every rectifiable path not in Γ. To prove this result we need the following assumptions: the exponent satisfies $p\colon \Omega \to [m, M]$ for $1 < m \le M < \infty$ and smooth functions are dense in Sobolev space.

1. Introduction

Variable exponent Lebesgue and Sobolev spaces have attracted steadily increasing interest over the last couple of years. These spaces have been independently discovered by several researchers [9, 15, 19, 22]. These investigators were motivated by differential equations with non-standard coercivity conditions, arising for instance from modeling certain fluids (e.g, [1, 7, 18]). For some of the latest advances in the study of variable exponent spaces see [3, 4, 11, 12, 16].

Classical Sobolev spaces can be characterized in many different ways. The ACL-characterization is that a function belongs to the Sobolev space $W^{1,p}$ if and only if it has a Lebesgue p-integrable representative which is absolutely continuous on almost every line segment parallel to the coordinate axes and the classical derivatives defined almost everywhere are Lebesgue p-integrable. Note that the classical derivatives of the

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representative coincide with the distributional derivatives almost everywhere. This characterization goes back to Nikodym [17].

A finer version of the ACL-characterization is the ACC-characterization. In the ACC-characterization, we consider not only line segments but all rectifiable curves. Fuglede proved in [10] that a Sobolev function in $W^{1,p}$ has a representative which is absolutely continuous on every curve not belonging to a certain exceptional family of zero p-modulus. On the other hand, it is clear that a function which has the ACC-property also has the ACL-property.

In this article we generalize these results to the variable exponent Sobolev spaces. We patch together pieces of classical proofs in a way which neatly avoids all the properties not possessed by variable exponent spaces. For instance the original proof by Fuglede [10] relies on convolutions and translation invariance which cannot used in our context.

The ACL-characterization follows immediately from the fact that variable exponent Sobolev space can be locally imbedded in $W^{1,1}$. For the ACC-characterization we use the variable exponent Sobolev capacity and quasi-continuous representatives considered in the variable exponent setting by Harjulehto, Hästö, Koskenoja and Varonen in [12]. We first show that the well-known relation between modulus and capacity holds, i.e. if a set has zero capacity, then it defines a curve family with zero modulus. These results are derived under the assumption that smooth functions are dense in Sobolev space. Although this is not always the case, it is quite a lenient condition, allowing for instance some discontinuous exponents (see Section 3).

DEFINITIONS. By Ω we always denote an open subset of $\mathbb{R}^n$. Let $p : \Omega \to [1, \infty)$ be a measurable function, called the *variable exponent* on Ω, and set $p^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\Omega} p(x)$. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which there exists $\lambda > 0$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)}\, dx < \infty$. We define the *Luxemburg norm* on this space by

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}.$$

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the subspace of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient exists almost everywhere and satisfies $|\nabla u| \in L^{p(\cdot)}(\Omega)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\Omega)$ a Banach space. One central property of these spaces is that $\varrho_{p(\cdot)}(u_i) \to 0$ if and only if $\|u_i\|_{p(\cdot)} \to 0$, provided that $p^+ < \infty$. This and many other basic results are proved in [15].

2. The modulus

A curve γ in $\mathbb{R}^n$ is a non-constant continuous map $\gamma : I \to \mathbb{R}^n$, where $I = [a, b]$ is a closed interval in $\mathbb{R}$. The image of γ, $\gamma(I)$, is denoted by $|\gamma|$. Let Γ_{rect} be the family of rectifiable curves in $\mathbb{R}^n$. Note that if the curve γ is rectifiable, then we can assume that $I = [0, \ell(\gamma)]$, where $\ell(\gamma)$ is the length of γ.

Given a family Γ of rectifiable curves, we denote by $F(\Gamma)$ the set of all *admissible functions*, i.e. all Borel functions $u : \mathbb{R}^n \to [0, \infty]$ such that

$$\int_\gamma u\, ds \geq 1$$

for every $\gamma \in \Gamma$, where ds represents integration with respect to curve length. We define *the $p(\cdot)$-modulus of* Γ by

$$\mathrm{M}_{p(\cdot)}(\Gamma) = \inf_{u \in F(\Gamma)} \int_{\mathbb{R}^n} u(x)^{p(x)} dx.$$

If $F(\Gamma) = \emptyset$, then we set $\mathrm{M}_{p(\cdot)}(\Gamma) = \infty$.

Lemma 2.1

The $p(\cdot)$-modulus is an outer measure on the space of all curves of $\mathbb{R}^n$. This means that the following hold:

(1) $\mathrm{M}_{p(\cdot)}(\emptyset) = 0$,
(2) $\Gamma_1 \subset \Gamma_2$ *implies* $\mathrm{M}_{p(\cdot)}(\Gamma_1) \leq \mathrm{M}_{p(\cdot)}(\Gamma_2)$,
(3) $\mathrm{M}_{p(\cdot)}\left(\bigcup_{i=1}^\infty \Gamma_i\right) \leq \sum_{i=1}^\infty \mathrm{M}_{p(\cdot)}(\Gamma_i)$.

Proof. Since the zero function belongs to $F(\emptyset)$ we obtain (1). If $\Gamma_1 \subset \Gamma_2$ then $F(\Gamma_1) \supset F(\Gamma_2)$ and hence (2) holds.

To prove (3) we may assume that the sum in the right-hand side is finite. For $\varepsilon > 0$ we pick $u_i \in F(\Gamma_i)$ such that

$$\int_{\mathbb{R}^n} u_i(x)^{p(x)} dx < \mathrm{M}_{p(\cdot)}(\Gamma_i) + \varepsilon 2^{-i}.$$

We set $u(x) = \left(\sum_{i=1}^\infty u_i(x)^{p(x)}\right)^{\frac{1}{p(x)}}$. Since $u \geq u_i$ for all i, the function u satisfies $\int_\gamma u\, ds \geq 1$ for every $\gamma \in \bigcup_{i=1}^\infty \Gamma_i$. Thus we obtain

$$\mathrm{M}_{p(\cdot)}\left(\bigcup_{i=1}^\infty \Gamma_i\right) \leq \int_{\mathbb{R}^n} u(x)^{p(x)} dx = \int_{\mathbb{R}^n} \sum_{i=1}^\infty u_i(x)^{p(x)} dx \leq \sum_{i=1}^\infty \mathrm{M}_{p(\cdot)}(\Gamma_i) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain (3). $\square$

A family of curves Γ is said to be *exceptional* if $\mathrm{M}_{p(\cdot)}(\Gamma) = 0$. The following lemma is a generalization of [10, Theorem 3(f)].

Lemma 2.2 (Fuglede's Lemma)

Let $(u_i)_{i=1}^\infty$ be a sequence of non-negative Borel functions in $L^{p(\cdot)}(\mathbb{R}^n)$ converging to zero. Then there exists a subsequence $(u_{i_k})_{k=1}^\infty$ and an exceptional set Γ of rectifiable curves such that for all rectifiable $\gamma \notin \Gamma$ we have

$$\lim_{k \to \infty} \int_\gamma u_{i_k} ds = 0.$$

Proof. We choose a subsequence of $(u_i)_{i=1}^\infty$, which is again denoted by $(u_i)_{i=1}^\infty$, so that

$$\|u_i\|_{p(\cdot)} \le \frac{2^{-i}}{i}. \tag{2.3}$$

Let Γ be the family of curves such that for all $\gamma \in \Gamma$

$$\int_\gamma u_i\, ds \not\to 0,$$

as $i \to \infty$. We write

$$\Gamma_i = \left\{ \gamma \in \Gamma : \int_\gamma u_i\, ds \ge \frac{1}{i} \right\}$$

and

$$v = \sum_{i=1}^\infty i u_i.$$

Observe that iu_i is admissible for Γ_i and that v is a non-negative Borel function in $\mathbb{R}^n$. If $\gamma \in \Gamma$, then $\gamma \in \Gamma_i$ for infinitely many i and hence by the Lebesgue monotone convergence theorem

$$\int_\gamma v\, ds = \sum_{i=1}^\infty \int_\gamma i u_i\, ds = \infty.$$

But this means that $v_k = \frac{v}{k}$, $k = 1, 2, \ldots$, is admissible for Γ. On the other hand, it follows from (2.3) and the homogeneity of the norm that

$$\|v_k\|_{p(\cdot)} = \frac{1}{k}\|v\|_{p(\cdot)} \le \frac{1}{k} \sum_{i=1}^k \|iu_i\|_{p(\cdot)} \le \frac{1}{k}$$

for each $k = 1, 2, \ldots$. This means that $v_k \to 0$ in $L^{p(\cdot)}$ and for every $k = 1, 2, \ldots$

$$\mathrm{M}_{p(\cdot)}(\Gamma) \le \int_{\mathbb{R}^n} (v_k(x))^{p(x)}\, dx \le \|v_k\|_{p(\cdot)} \le \frac{1}{k},$$

where the second inequality is [15, (2.11)]. Letting $k \to \infty$ we conclude that $\mathrm{M}_{p(\cdot)}(\Gamma) = 0$, as required. □

3. The capacity

Next we consider a variable exponent Sobolev capacity, which was introduced by Harjulehto, Hästö, Koskenoja and Varonen in [12]. Suppose that E is an arbitrary subset of $\mathbb{R}^n$. The *Sobolev* $p(\cdot)$*-capacity* of E is defined by

$$C_{p(\cdot)}(E) = \inf \int_{\mathbb{R}^n} \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) dx,$$

where the infimum is taken over those $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ which are at least 1 in some open set containing E. If $1 < p^- \le p^+ < \infty$, then the Sobolev $p(\cdot)$-capacity is an outer measure and a Choquet capacity [12, Corollaries 3.3 and 3.4]. As in the fixed exponent case, the capacity is a finer measure than the n-dimensional Lebesgue measure [12, Section 4]. For example, if $C_{p(\cdot)}(E) = 0$, then the s-dimensional Hausdorff measure of E is zero for every $s > n - p^-$ (if $p^- > n$, then the set E is empty).

We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is *quasi-continuous* if for every $\varepsilon > 0$ there exists an open set $\mathcal{O}$ with $C_{p(\cdot)}(\mathcal{O}) < \varepsilon$ such that u restricted to $\mathbb{R}^n \setminus \mathcal{O}$ is continuous. We recall the following fact [12, Theorem 5.2], which is crucial to us here. Assume that $1 < p^- \le p^+ < \infty$ and the class of continuous functions is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Then every $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ has a quasicontinuous representative, i.e. there exists a quasi-continuous function $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u = v$ almost everywhere in $\mathbb{R}^n$.

For our remaining results we need to assume that continuously differentiable functions are dense in Sobolev space. This is a real assumption, since it is well-known that this is not always the case in the variable exponent setting. We only have some partial results:

(i) Samko proved in [20] that smooth function are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$ if $p^+ < \infty$ and there exists a constant $C > 0$ such that for every $|x - y| \le \frac{1}{2}$ the exponent p satisfies

$$|p(x) - p(y)| \le \frac{C}{-\log|x - y|}.$$

Diening proved a similar, though slightly weaker, result [2].

(ii) Edmunds and Rákosník showed that a certain monotonicity condition on the exponent is also sufficient for the density of smooth functions, see [5].

(iii) Hästö [14] gave an example of the variable exponent Sobolev space in which continuous functions are not dense. In this example the exponent has growth just slightly greater than allowed in (i) at a saddle point. Therefore the previous two conditions together seem to be quite close to optimal.

Shanmugalingam proved the following lemma in a metric measure space with a fixed exponent, [21, Lemma 3.6]. Our proof uses the same idea. We denote by Γ_E the family of all rectifiable curves whose image intersect the set E.

Lemma 3.1

Assume that $1 < p^- \le p^+ < \infty$ and C^1-functions are dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$. Suppose that $E \subset \mathbb{R}^n$. If $C_{p(\cdot)}(E) = 0$, then $\mathrm{M}_{p(\cdot)}(\Gamma_E) = 0$.

Proof. Since $C_{p(\cdot)}(E) = 0$ we can choose a function $u_i \in W^{1,p(\cdot)}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$, for every i, such that $\|u_i\|_{1,p(\cdot)} \le 2^{-i}$ and $u_i(x) \ge 1$ for every $x \in E$. We define

$$v_k = \sum_{i=1}^{k} |u_i|.$$

For every $l \ge m$ we find that

$$\|v_l - v_m\|_{1,p(\cdot)} \le \sum_{i=m+1}^{l} \|u_i\|_{1,p(\cdot)} \le 2^{-m}$$

and therefore the sequence $(v_k)_{k=1}^{\infty}$ is a Cauchy sequence in the Banach space $W^{1,p(\cdot)}(\mathbb{R}^n)$. Since the sequence $v_k(x)$ is non-negative and increasing for every $x \in \mathbb{R}^n$ the limit $v(x) = \lim_{k\to\infty} v_k(x)$ (possibly $+\infty$) exists for every $x \in \mathbb{R}^n$ and $v \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is a Borel function.

For $x \in E$ we see that $v_k(x) \ge k$ for every k and thus

$$E \subset E_\infty = \{x \in \mathbb{R}^n \colon \lim_{k\to\infty} v_k(x) = \infty\}.$$

Therefore it suffices to show that $\mathrm{M}_{p(\cdot)}(\Gamma_{E_\infty}) = 0$.

Since the ∇v_k are continuous, ∇v is a Borel function, and so Lemma 2.2 gives a subsequence of (v_k), denoted again by (v_k), such that there is an exceptional family Γ_1 and

$$\lim_{k\to\infty} \int_\gamma |\nabla v_k - \nabla v| \, ds = 0 \tag{3.2}$$

for every rectifiable curve γ not in Γ_1.

Let Γ_2 be the family of all curves γ such that $\int_\gamma v \, ds = \infty$ and Γ_3 the family of curves γ with $\int_\gamma |\nabla v| \, ds = \infty$. Since v/i is admissible for Γ_2 and every $i = 1, 2, 3 \ldots$ and since $v \in L^{p(\cdot)}(\mathbb{R}^n)$, we find that

$$\mathrm{M}_{p(\cdot)}(\Gamma_2) \le \int_{\mathbb{R}^n} \Big(\frac{v(x)}{i}\Big)^{p(x)} + \Big(\frac{|\nabla v(x)|}{i}\Big)^{p(x)} dx \le \frac{\|v\|_{1,p(\cdot)}}{i}$$

for large enough i, by [15, (2.11)]. Therefore $\mathrm{M}_{p(\cdot)}(\Gamma_2) = 0$. Similarly $\mathrm{M}_{p(\cdot)}(\Gamma_3) = 0$ and hence by subadditivity $\mathrm{M}_{p(\cdot)}(\Gamma^*) = 0$, where $\Gamma^* = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. To complete the proof we show that $\Gamma_{E_\infty} \subset \Gamma^*$. Fix $\gamma \in \Gamma_{E_\infty}$ and suppose that $\gamma \notin \Gamma^*$. Since $\gamma \notin \Gamma_2$ there is $y \in |\gamma|$ with $v(y) < \infty$. For any point $x \in |\gamma|$ we find that

$$|v_i(x)| \le |v_i(x) - v_i(y)| + |v_i(y)| \le \int_\gamma |\nabla v_i| \, ds + |v_i(y)|,$$

because $v_i \circ \gamma$ is absolutely continuous on $[0, \ell(\gamma)]$. Using (3.2) and $\gamma \notin \Gamma_1$ for the first inequality and $\gamma \notin \Gamma_3$ for the second inequality we find that

$$\limsup_{i\to\infty} |v_i(x)| \le \limsup_{i\to\infty} |v_i(y)| + \int_\gamma |\nabla v| \, ds < \infty.$$

Hence $v(x) < \infty$ for all $x \in |\gamma|$ and $\gamma \notin \Gamma_{E_\infty}$. Thus $\Gamma_{E_\infty} \subset \Gamma^*$, which completes the proof. □

4. Fuglede's Theorem

We say that $u \colon \Omega \to \mathbb{R}$ is *absolutely continuous on lines*, $u \in ACL(\Omega)$, if u is absolutely continuous on almost every line segment in Ω parallel to the coordinate axes. Note that an ACL function has classical derivatives almost everywhere.

An ACL function is said to belong to $ACL^{p(\cdot)}(\Omega)$ if $|\nabla u| \in L^{p(\cdot)}(\Omega)$. Since $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ locally, we obtain the following lemma by [8, Chapter 4.9] or [23, Theorem 2.1.4].

Lemma 4.1

If $u \in ACL^{p(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$, then it has classical partial derivatives almost everywhere and these coincide with the weak partial derivatives as distributions so that $u \in W^{1,p(\cdot)}(\Omega)$. If $u \in W^{1,p(\cdot)}(\Omega)$, then there exists $v \in ACL^{p(\cdot)}(\Omega)$ such that $u = v$ almost everywhere. In short, $ACL^{p(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega)$.

Let $u\colon \mathbb{R}^n \to \mathbb{R}$ and let Γ be the family of curves γ parametrized by arc-length such that $u\circ\gamma$ is not absolutely continuous on $[0, \ell(\gamma)]$. We say that u is *absolutely continuous on curves*, $u \in ACC_{p(\cdot)}(\Omega)$, if $\mathrm{M}_{p(\cdot)}(\Gamma) = 0$. It is clear that $ACC_{p(\cdot)}(\Omega) \subset ACL(\Omega)$. An $ACC_{p(\cdot)}(\Omega)$ function is said to belong to $ACC^{p(\cdot)}(\Omega)$ if $|\nabla u| \in L^{p(\cdot)}(\Omega)$.

The following theorem generalizes a result of Fuglede, [10, Theorem 14], to the variable exponent case.

Theorem 4.2 (Fuglede's Theorem)

Assume that $1 < p^- \le p^+ < \infty$ *and that* C^1*-functions are dense in* $W^{1,p(\cdot)}(\mathbb{R}^n)$. *Then*

$$ACC^{p(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n) = W^{1,p(\cdot)}(\mathbb{R}^n).$$

Proof. By Lemma 4.1, $ACL^{p(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n) \subset W^{1,p(\cdot)}(\mathbb{R}^n)$ and hence

$$ACC^{p(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n) \subset W^{1,p(\cdot)}(\mathbb{R}^n).$$

To prove the converse, fix $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$. By [12, Lemma 5.1] there exists a sequence (u_i) of functions in $W^{1,p(\cdot)}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ such that $u_i \to u$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$ and $u_i(x) \to \tilde{u}(x)$ for every $x \in \mathbb{R}^n$ except in the set E of $p(\cdot)$-capacity zero. Here $\tilde{u}$ is a $p(\cdot)$-quasi-continuous representative of u.

Since $u_i \to \tilde{u}$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$ we may assume, passing to a subsequence, that

$$\|\nabla u_{i+1} - \nabla u_i\|_{p(\cdot)} \le 2^{-i}, \tag{4.3}$$

for every $i = 1, 2, \ldots$. Since

$$u_i = u_1 + \sum_{j=1}^{i-1} (u_{j+1} - u_j),$$

we have $|\nabla u_i| \le g_i$ for every $i = 1, 2, \ldots$, where

$$g_i = |\nabla u_1| + \sum_{j=1}^{i-1} |\nabla(u_{j+1} - u_j)|.$$

The sequence (g_i) is increasing, hence the limit (possibly ∞) $g(x) = \lim_{i\to\infty} g_i(x)$ exists for each $x \in \mathbb{R}^n$. Since the g_i's are continuous, g is a Borel function and it follows from (4.3) that $g_i \to g$ in $L^{p(\cdot)}(\mathbb{R}^n)$.

Let Γ_1 be the family of all rectifiable curves γ in $\mathbb{R}^n$ such that

$$\int_\gamma g\, ds = \infty.$$

Since g/j is admissible for Γ_1, we find that $\mathrm{M}_{p(\cdot)}(\Gamma_1) = 0$. By Lemma 2.2 we choose a subsequence of (g_i) such that

$$\limsup_{i\to\infty} \int_\gamma |g_i - g|\, ds = 0$$

except in an exceptional set Γ_2. Since the $p(\cdot)$-capacity of E is zero, Lemma 3.1 implies that $\mathrm{M}_{p(\cdot)}(\Gamma_E) = 0$. We write $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_E$. By subadditivity, $\mathrm{M}_{p(\cdot)}(\Gamma) = 0$. The curve family Γ has the following special property: if $\gamma_1 \in \Gamma$ and γ_2 is a curve containing

γ_1, i.e. $|\gamma_1| \subset |\gamma_2|$, then $\gamma_2 \in \Gamma$. This property follows, since each of Γ_1, Γ_2 and Γ_E has it.

It remains to show that $\tilde{u}$ is absolutely continuous on each rectifiable curve $\gamma : [0, l(\gamma)] \to \mathbb{R}^n$, $\gamma \notin \Gamma$. Let (a_j, b_j), $j = 1, 2, \ldots, k$, be disjoint intervals on $[0, l(\gamma)]$ and write $A = \cup_{j=1}^k [a_j, b_j]$. Then

$$
\begin{aligned}
\sum_{j=1}^{k} |\tilde{u}(\gamma(b_j)) - \tilde{u}(\gamma(a_j))| &= \lim_{i\to\infty} \sum_{j=1}^{k} |u_i(\gamma(b_j)) - u_i(\gamma(a_j))| \\
&\le \limsup_{i\to\infty} \sum_{j=1}^{k} \int_{\gamma|_{[a_j,b_j]}} |\nabla u_i|\, ds \\
&\le \lim_{i\to\infty} \int_{\gamma|_A} g_i\, ds = \int_{\gamma|_A} g\, ds.
\end{aligned}
\tag{4.4}
$$

The first equality follows, since γ does not intersect E, and convergence is pointwise outside E. The first inequality follows, since every u_i is absolutely continuous. The second inequality follows, since $|\nabla u_i| \le g_i$, and the last equality follows, since $\gamma|_{[a_j,b_j]} \notin \Gamma$ for any j. Since $g \circ \gamma \in L^1([0, l(\gamma)])$, inequality (4.4) implies that $\tilde{u} \circ \gamma$ is absolutely continuous on $[0, l(\gamma)]$ as required. The proof is complete. □

Remark 4.5. In summary, Lemma 4.1 and Theorem 4.2 imply that

$$ACC^{p(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n) = ACL^{p(\cdot)}(\mathbb{R}^n) \cap L^{p(\cdot)}(\mathbb{R}^n) = W^{1,p(\cdot)}(\mathbb{R}^n).$$

It is also interesting to note that the $p(\cdot)$-quasi-continuous representative of $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ is the representative which is absolutely continuous on every curve except on a set of zero $p(\cdot)$-modulus.

Theorem 4.2 is formulated in $\mathbb{R}^n$ and not in an open subset of $\mathbb{R}^n$. However, since the $ACC^{p(\cdot)}$-condition has local character, we indicate briefly how the corresponding problem can be handled in an open set $\Omega \subset \mathbb{R}^n$.

If $p : \Omega \to [1, \infty)$, then the $p(\cdot)$-modulus of a curve family Γ of curves in Ω is defined by

$$\mathrm{M}_{p(\cdot)}(\Gamma) = \inf_{u \in F(\Gamma)} \int_{\Omega} u(x)^{p(x)} dx.$$

where $F(\Gamma)$ is as in Section 2. Note that we can always take $u = 0$ in $\mathbb{R}^n \setminus \Omega$.

The extension of a function $u \in W^{1,p(\cdot)}(\Omega)$ to a function $\tilde{u} \in W^{1,\tilde{p}(\cdot)}(\mathbb{R}^n)$ involves an extension of p to $\tilde{p}$: $\mathbb{R}^n \to [1, \infty)$. In general this can be quite difficult, see [6, Theorem 4.1] and [3, Section 4]. However, if $u \in W^{1,p(\cdot)}(\Omega)$ has compact support in Ω, then this can be done easily.

Theorem 4.6 (Fuglede's Theorem in Ω)

Suppose that $p : \Omega \to [1, \infty)$ and $1 < p^- \le p^+ < \infty$. If $C_0^1(\Omega)$-functions are dense in the set of functions in $W^{1,p(\cdot)}(\Omega)$, with compact support in Ω, then

$$ACC^{p(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega) = ACL^{p(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega).$$

Proof. Let $u \in W^{1,p(\cdot)}(\Omega)$ and let U be an open set such that $\overline{U}$ is a compact subset of Ω. We first show that $u|_U$ has a representative which is in $ACC^{p(\cdot)}(U)$. Choose a function $\xi \in C_0^\infty(\Omega)$ such that $0 \le \xi \le 1$ and $\xi = 1$ on U. Now $u\xi$ has compact support in Ω. We define an extension of p by

$$\tilde{p}(x) = \begin{cases} p(x) & , \quad x \in \Omega \\ 2 & , \quad x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Setting $u\xi = 0$ outside Ω we find that

$$\|u\xi\|_{1,\tilde{p}(\cdot)} = \|u\xi\|_{1,p(\cdot)}$$

where the first norm is taken in $\mathbb{R}^n$ and the second in Ω. As in the proof of Theorem 4.2 we conclude that $u\xi$ has a representative $\tilde{u}$ such that $\tilde{u} \in ACC^{\tilde{p}(\cdot)}(\mathbb{R}^n)$. Since $p(x) = \tilde{p}(x)$ in U, $\tilde{u}|_U$ is a representative of $u|_U$ in $W^{1,p(\cdot)}(U)$ and $\tilde{u}|_U$ is also in $ACC^{p(\cdot)}(U)$.

Let Γ be the family of curves on which u is not absolutely continuous. Let U_i be an increasing sequence of open sets whose closures lie in Ω such that $\cup U_i = \Omega$. Let Γ_i be the family of curves in Γ which lie wholly in U_i. Then $\Gamma = \cup \Gamma_i$. By the previous argument, $\mathrm{M}_{p(\cdot)}(\Gamma_i) = 0$, hence by subadditivity $\mathrm{M}_{p(\cdot)}(\Gamma) = 0$. This completes the proof. $\square$

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