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Willmore submanifolds in the unit sphere

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Abstract

In this paper we generalize the self-adjoint differential operator (used by Cheng-Yau) on hypersurfaces of a constant curvature manifold to general submanifolds. The generalized operator is no longer self-adjoint. However we present its adjoint operator. By using this operator we get the pinching theorem on Willmore submanifolds which is analogous to the pinching theorem on minimal submanifold of a sphere given by Simon and Chern-Do Carmo-Kobayashi.

§0. Introduction

Let M be an n-dimensional manifold isometrically immersed in sphere S^{n+p} of dimension $n+p$. Let h be the second fundamental form of this submanifold. We denote by S the square of the length of h, by $\vec{H}$ the mean curvature vector, by $|\vec{H}|$ the length of $\vec{H}$ respectively. We define a nonnegative function ρ^2 by

$$\rho^2 = S - n|\vec{H}|^2 \tag{0.1}$$

The Willmore functional W is defined by

$$W(M) = \int_M \rho^n dM \tag{0.2}$$

which is a conformal invariant under Möbius (or conformal) transformations of S^{n+p} (see [2], [9], [10]). Recently, Changping Wang got the Euler-Lagrange equations in [9], and Zhen Guo, Haizhong Li and Changping Wang got the second variation formula in the framework of Möbius geometry [6]. At the same time, in [6], the authors gave

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Euler-Lagrange equations with Euclidean quantities as follows

$$\begin{aligned} &-\rho^{n-2}[SH^\alpha + H^\beta h^\beta_{ij} h^\alpha_{ij} - h^\alpha_{ij} h^\beta_{ik} h^\beta_{kj} - n|\vec{H}|^2 H^\alpha] \\ &\qquad + (n-1)\Delta(\rho^{n-2}H^\alpha) - (\rho^{n-2})_{,ij}(nH^\alpha\delta_{ij} - h^\alpha_{ij}) = 0, \end{aligned} \tag{0.3}$$

where h^α_{ij} are the components of h with respect to a local orthonormal frame

$$\{e_i, e_\alpha; 1 \le i \le n, n+1 \le n+p\}$$

(e_i is tangent to M and e_α is normal to M) and $H^\alpha = \frac{1}{n}\sum_i h^\alpha_{ii}$. In particular, when $p = 1$ and $n = 2$, equation (0.3) reduces to the well-known form

$$\Delta H + 2H(H^2 - K) = 0, \tag{0.4}$$

where H and K are mean curvature and Gauss curvature. A Submanifold is called *Willmore submanifold* if it satisfied equation (0.3). It is easy to see from (0.4) that all minimal surfaces are Willmore surfaces. The nonminimal Willmore surfaces exist in large quantities (see [1], [5] and [7]). However, in case $n \ge 3$, there are minimal submanifolds which are not Willmore submanifolds. For instance, Clifford minimal hypersurfaces $M_k = S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ are not Willmore hypersurfaces if $2k \ne n$(cf. [6]). In [6], we proved that tori

$$W^n_k = S^k\left(\sqrt{\frac{n-k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{k}{n}}\right) \tag{0.5}$$

are Willmore hypersurfaces and are stable. We call W^n_k *Willmore tori.* It should be shown that Veronese surface and Willmore tori satisfy $\rho^2 = n/(2 - 1/p)$.

In this paper we characterize the tow Willmore submanifolds by using Euclidean invariant ρ^2. Our main result is stated as follows:

Main Theorem. *Let M be an n-dimensional compact oriented Willmore submanifold in an $(n+p)$-dimensional unit sphere, without umbilical point. Then*

$$\int_M \left(\left(2 - \frac{1}{p}\right)\rho^2 - n\right)\rho^n dM \ge 0. \tag{0.6}$$

In particular, if $\rho^2 \le n/(2-1/p)$, then $\rho^2 = n/(2-1/p)$, and M is isometric to either

(i) Willmore tori W^n_k in S^{n+1}. or

(ii) Verones surface in S^4.

We organize this paper as follows. For the purpose to prove main Theorem, we define the operator $\Box$ and the operator $\Box^*$ in §1, and prove they are adjoint with respect to suitable inner product. *It is very interesting that this operator appears naturally in the equation satisfied by Willmore submanifolds.* In §2 we present key lemmas and formulas. In §3 we prove main Theorem.

§1. The operator $\Box$ and its adjoint operator

Let M be an n-dimensional submanifold isometrically immersed in space form $N^{n+p}(c)$ with constant sectional curvature c. For each point $P \in M$, we choose a local orthonormal frame field $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$ around P, such that $e_1, \cdots, e_n$ are tangent to M. The corresponding dual frame field is denoted by $\{\omega_1, \cdots, \omega_n, \omega_{n+1}, \cdots, \omega_{n+p}\}$. When restricted on M, $\omega_\alpha = 0$. We make the following convention on the range of indices:

$$1 \leq i, j, k, \cdots \leq n; n+1 \leq \alpha, \beta, \gamma \cdots \leq n+p,$$

and shall agree that repeated indices are summed over the respective ranges. Let $h = h^\alpha_{ij}\omega_i \otimes \omega_j e_\alpha$ denote the second fundamental form and $\vec{H} = \sum_\alpha \frac{1}{n}(\sum_i h^\alpha_{ii})e_\alpha$ the mean curvature vector of M to N. Then we have Gauss equation

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{1.1}$$

Codazzi equation

$$h^\alpha_{ij,k} - h^\alpha_{ik,j} = 0, \tag{1.2}$$

and Ricci equation

$$h^\alpha_{ij,kl} - h^\alpha_{ij,lk} = h^\alpha_{im}R_{mjkl} + h^\alpha_{mj}R_{mikl} + h^\beta_{ij}R_{\beta\alpha kl}, \tag{1.3}$$

where

$$R_{\alpha\beta ij} = h^\alpha_{ik}h^\beta_{kj} - h^\beta_{ik}h^\alpha_{kj}. \tag{1.4}$$

For a section $\xi^\alpha e_\alpha$ of the normal bundle $T^\perp(M)$ we define the covariant derivative $\xi^\alpha_{,i}$ of ξ^α by

$$\xi^\alpha_{,i}\omega_i = d\xi^\alpha + \xi^\beta\omega_{\beta\alpha}, \tag{1.5}$$

and the covariant derivative $\xi^\alpha_{,ij}$ of $\xi^\alpha_{,i}$ by

$$\xi^\alpha_{,ij}\omega_j = d\xi^\alpha_{,i} + \xi^\alpha_{,j}\omega_{ji} + \xi^\beta_{,i}\omega_{\beta\alpha}, \tag{1.6}$$

where ω_{ij} and $\omega_{\alpha\beta}$ denote the connection forms on M and $T^\perp(M)$, respectively.

For a section $\phi = \phi^\alpha_{ij}\omega_i\omega_j e_\alpha$ of the vector bundle $T^\perp(M) \otimes T^*(M) \otimes T^*(M)$ we can define its covariant derivative ϕ^α_{ijk} by

$$\phi^\alpha_{ij,k}\omega_k = d\phi^\alpha_{ij} + \phi^\alpha_{ik}\omega_{kj} + \phi^\alpha_{kj}\omega_{ki} + \phi^\beta_{ij}\omega_{\beta\alpha}. \tag{1.7}$$

We denote the set of the smooth sections of normal bundle $T^\perp(M)$ by $C^\infty(T^\perp(M))$ and the set of the smooth functions of M by $C^\infty(M)$, and for each ϕ, define the operator

$$\Box^*_\phi : \quad C^\infty(T^\perp(M)) \quad \rightarrow \quad C^\infty(M)$$

by

$$\Box^*_\phi \xi = \sum_{\alpha,i,j} \phi^\alpha_{ij}\xi^\alpha_{,ij}, \tag{1.8}$$

where $\xi = \xi^\alpha e_\alpha$ is a section of $T^\perp(M)$. Since $\sum_{\alpha,i,j}\phi^\alpha_{ij}\xi^\alpha_{,ij}$ can be viewed as the inner product $<\phi, \nabla^2\xi>$ of tow tensors ϕ and $\nabla^2\xi = \xi^\alpha_{,ij}\omega_i\omega_j e_\alpha$ in $C^\infty(T^\perp M \otimes T^*M \otimes T^*M)$, the quantity is independence of the choice of the local orthonormal $\{e_i, e_\alpha\}$.

We can also define another operator

$$\Box_\phi : \quad C^\infty(M) \quad \rightarrow \quad C^\infty(T^\perp M)$$

by

$$\Box_\phi f = \sum_{\alpha,i,j} \phi^\alpha_{ij} f_{,ij} e_\alpha. \tag{1.9}$$

For any point $q \in M$, let $<\,,\,>_q$ denote the inner product on $T_q^{\perp}M$ (the fiber of $T^{\perp}M$) deduced by the metric of N. Then for any $\xi, \eta \in C^{\infty}(T^{\perp}M)$, since $\xi_q, \eta_q \in T_q^{\perp}M$, we can define a function $<\xi, \eta> \in C^{\infty}(M)$ by $<\xi, \eta>(q) = <\xi_q, \eta_q>_q$, and so can define the global inner product $(\ ,\ )$ on $C^{\infty}(T^{\perp}M)$ by

$$(\xi, \eta) = \int_M <\xi, \eta> dM. \tag{1.10}$$

Let the same symbol $(\ ,\ )$ denote the L^2-inner product. Then we have.

Theorem 1.1

Let M be a compact oriented submanifold. If ϕ satisfies the conditions

$$(i)\ \ \phi_{ij}^{\alpha} = \phi_{ji}^{\alpha}, \qquad (ii)\ \ \sum_j \phi_{ij,j}^{\alpha} = 0,$$

then $\Box_\phi$ and $\Box_\phi^*$ are adjoint, which means

$$(\Box_\phi^*\xi, f) = (\xi, \Box_\phi f). \tag{1.11}$$

Proof. By a direct computation we have

$$\begin{aligned} f\Box_\phi^*\xi = \sum_{\alpha,i,j} f\phi_{ij}^{\alpha}\xi_{ij}^{\alpha} &= \sum_j \Big(f\sum_{\alpha,i}\phi_{ij}^{\alpha}\xi_i^{\alpha}\Big)_{,j} - \sum_i\Big(\sum_{\alpha,j} f_{,j}\phi_{ij}^{\alpha}\xi^{\alpha}\Big)_{,i} \\ &+ \sum_{\alpha,i,j} f_{,ij}\phi_{ij}^{\alpha}\xi^{\alpha} + \sum_{\alpha,i,j} f_{,j}\phi_{ij,i}^{\alpha}\xi^{\alpha} - \sum_{\alpha,i,j} f\phi_{ij,j}^{\alpha}\xi_{,i}^{\alpha}. \end{aligned} \tag{1.12}$$

Substituting the conditions (i) and (ii) into (1.12) and making use of Green's Theorem, we get

$$\int_M f\Box_\phi^*\xi dM = \int_M <\Box_\phi f, \xi> dM.$$

This completes the proof of (1.11).

Now we take

$$\phi_{ij}^{\alpha} = mH^{\alpha}\delta_{ij} - h_{ij}^{\alpha}, \tag{1.13}$$

then this ϕ satisfies the conditions in Theorem 1.1 by Codazzi equation (1.2). Let $\Box$ and $\Box^*$ denote the operators corresponding to ϕ defined by (1.8) and (1.9). Then

$$(\Box^*\xi, f) = (\xi, \Box f) \tag{1.14}$$

and

$$\int_M \Box^*\xi = 0 \tag{1.15}$$

holds. □

Remark. In case $p = 1$, the operator $\Box$ is essentially the operator given by Cheng and Yau in [3]. The only difference is: the operator acts on $C^{\infty}(M)$, not on the set of the sections of $T^{\perp}M$. However, in this case, we have a bijective mapping $C^{\infty}(T^{\perp}M) \ni \xi = fe_{n+1} \leftrightarrow f \in C^{\infty}(M)$. By substituting f for fe_{n+1}, we see that $\Box$ is exactly the operator defined by Cheng-Yau.

§2. The formulas and the lemmas

Let $\vec{H} = \sum_\alpha H^\alpha e_\alpha$, where $H^\alpha = \frac{1}{n}\sum_i h^\alpha_{ii}$. Then we have

$$\sum_k h^\alpha_{kk,i} = nH^\alpha_{,i}, \tag{2.1}$$

and

$$\sum_k h^\alpha_{kk,ij} = nH^\alpha_{,ij}. \tag{2.2}$$

It is easy to check that

$$h^\alpha_{ij}h^\alpha_{kk,ij} = -n\Box^*\vec{H} + n^2 H^\alpha \Delta H^\alpha, \tag{2.3}$$

where Δ denotes the Laplacian operator, which means that for a function $f, \Delta f = \sum_i f_{,ii}$; for a normal vector field $\xi^\alpha e_\alpha$, $\Delta\xi^\alpha = \sum_i \xi^\alpha_{,ii}$; and for tensor $h^\alpha_{ij}\omega_i\omega_j e_\alpha$, $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ij,kk}$.

On the other hand, we have

$$\frac{1}{2}\Delta|\vec{H}|^2 = \frac{1}{2}\Delta\sum_\alpha (H^\alpha)^2 = H^\alpha\Delta H^\alpha + |\nabla\vec{H}|^2, \tag{2.4}$$

where we denote the gradient operator by ∇ and define the normal $|\nabla\vec{H}|$ of $\nabla\vec{H} = H^\alpha_i\omega_i e_\alpha$ by

$$|\nabla\vec{H}|^2 = \sum_{i,\alpha}(H^\alpha_{,i})^2. \tag{2.5}$$

Remark. It should be noted that, in general, $|\nabla\vec{H}|^2 \neq |\nabla|\vec{H}||^2$.

From (2.3) and (1.3), we have

$$h^\alpha_{ij}\Delta h^\alpha_{ij} = -n\Box^*\vec{H} + n^2 H^\alpha\Delta H^\alpha + h^\alpha_{ij}h^\alpha_{km}R_{mijk} + h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} + h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}. \tag{2.6}$$

From (2.4) and (2.6) We have

$$\begin{aligned}\tfrac{1}{2}\Delta\rho^2 = {} & -n\Box^*\vec{H} + |\nabla h|^2 - n|\nabla\vec{H}|^2 + n(n-1)H^\alpha\Delta H^\alpha \\ & + \ h^\alpha_{ij}h^\alpha_{km}R_{mijk} + h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} + h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk},\end{aligned} \tag{2.7}$$

where $\rho^2 = S - n|\vec{H}|^2$. Noting that Δ is self-adjoint and $\Box^*$ and $\Box$ are adjoint, with respect to inner - product defined in Theorem 1.1, we get the following key formula

$$\begin{aligned}&\tfrac{1}{2}(\rho^{n-2}, \Delta\rho^2) = -n(\vec{H}, \Box\rho^{n-2}) + n(n-1)(\vec{H}, \Delta(\rho^{n-2}\vec{H})) \\ &\quad + (\rho^{n-2}, |\nabla h|^2 - n|\nabla\vec{H}|^2 + h^\alpha_{ij}h^\alpha_{km}R_{mijk} + h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} \quad + h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}).\end{aligned} \tag{2.8}$$

Remark. It is very interesting that the quantity

$$(\vec{H}, \Box\rho^{n-2}) - (n-1)(\vec{H}, \Delta(\rho^{n-2}\vec{H}))$$

appears naturally in the equation satisfied by Willmore submanifolds.

Lemma 2.1

$$|\nabla h|^2 - n|\nabla \vec{H}|^2 \geq 0, \tag{2.9}$$

and the equality holds if and only if $\nabla h = 0$.

Proof. Set a tensor F by

$$F^\alpha_{ijk} = h^\alpha_{ijk} - \frac{n}{n+2}\{H^\alpha_{,i}\delta_{jk} + H^\alpha_{,j}\delta_{ik} + H^\alpha_{,k}\delta_{ij}\}. \tag{2.10}$$

It is to check that

$$|F|^2 = \sum_{\alpha ijk}(F^\alpha_{ijk})^2 = |\nabla h|^2 - \frac{3n^2}{n+2}|\nabla \vec{H}|^2. \tag{2.11}$$

Thus, we have

$$|\nabla h|^2 - n|\nabla \vec{H}|^2 = |F|^2 + \frac{2(n-1)}{n+2}|\nabla \vec{H}|^2 \geq \frac{2(n-1)}{n+2}|\nabla \vec{H}|^2 \geq 0. \tag{2.12}$$

From (2.2) one can see that $|\nabla h|^2 - n|\nabla \vec{H}|^2 = 0$ implies $\nabla h = 0$. □

§3. Willmore submanifolds in unit sphere S^{n+p}

Let M^n be a submanifold of an $n+p$-dimensional unit sphere S^{n+p}. Willmore function $W(M)$ is defined by (0.2). Then M is a Willmore submanifold (it is a critical submanifold of the Willmore function $W(M)$) if and only if, for any α with $n+1 \leq \alpha \leq n+p$,

$$\begin{aligned} &- \rho^{n-2}[SH^\alpha + H^\beta h^\beta_{ij}h^\alpha_{ij} - h^\alpha_{ij}h^\beta_{ik}h^\beta_{kj} - n|\vec{H}|^2 H^\alpha] \\ &\quad + (n-1)\Delta(\rho^{n-2}H^\alpha) - (\rho^{n-2})_{,ij}(nH^\alpha\delta_{ij} - h^\alpha_{ij}) = 0. \end{aligned} \tag{3.1}$$

Multiplying this equation by H^α and taking the sum for all $\alpha's$, and then taking its integration we have

$$\begin{aligned} &(n-1)(\Delta(\rho^{n-2}\vec{H}), \vec{H}) - (\Box\rho^{n-2}, \vec{H}) \\ &= (\rho^{n-2}, [S - n|\vec{H}|^2]|\vec{H}|^2 + H^\beta h^\beta_{ij}h^\alpha_{ij}H^\alpha - H^\alpha h^\alpha_{ij}h^\beta_{ik}h^\beta_{kj}). \end{aligned} \tag{3.2}$$

Substituting (3.2) into (2.8) and using Lemma 2.1, we get the following theorem.

Theorem 3.1

Suppose M^n is a compact oriented Willmore submanifold in S^{n+p}. Then

$$\begin{aligned} 0 \geq \int_M \rho^{n-2}\{&n\rho^2|\vec{H}|^2 + nH^\beta h^\beta_{ij}h^\alpha_{ij}H^\alpha - nH^\alpha h^\alpha_{ij}h^\beta_{ik}h^\beta_{kj} \\ &+ h^\alpha_{ij}h^\alpha_{km}R_{mijk} + h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} + h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}\}dM. \end{aligned} \tag{3.3}$$

Set

$$Q = n\rho^2|\vec{H}|^2 + nH^\beta h_{ij}^\beta h_{ij}^\alpha H^\alpha - nH^\alpha h_{ij}^\alpha h_{ik}^\beta h_{kj}^\beta \\ +h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} + h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk}. \tag{3.4}$$

Let A_α denotes the matrix $(h_{ij}^\alpha)_{n\times n}$. From the Gauss equation, we have

$$Q = n\rho^2|\vec{H}|^2 + n\rho^2 + \sum_{\alpha,\beta} H^\alpha H^\beta tr(A_\alpha A_\beta) \\ -\sum_{\alpha,\beta} tr[(A_\alpha A_\beta - A_\beta A_\alpha)(A_\beta A_\alpha - A_\alpha A_\beta)] - \sum_{\alpha,\beta}(tr(A_\alpha A_\beta))^2. \tag{3.5}$$

Let

$$B_\alpha = A_\alpha - H^\alpha I,$$

where I is the unit matrix. Thus we can rewrite Q as follow

$$Q = n\rho^2|\vec{H}|^2 - nH^\alpha H^\beta tr(B_\alpha B_\beta) \\ +n\rho^2 - tr[(B_\alpha B_\beta - B_\beta B_\alpha)(B_\beta B_\alpha - B_\alpha B_\beta)] - \sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2. \tag{3.6}$$

Set

$$L_{\alpha\beta} = tr(B_\alpha B_\beta). \tag{3.7}$$

Then the $p \times p$ matrix $(L_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1}, \cdots, e_{n+p}$. We set

$$L_\alpha = L_{\alpha\alpha}, \tag{3.8}$$

then

$$\rho^2 = \sum_\alpha L_\alpha. \tag{3.9}$$

So we have

$$n\rho^2|\vec{H}|^2 - nH^\alpha H^\beta tr(B_\alpha B_\beta) = \sum_\alpha (H^\alpha)^2(\rho^2 - L_\alpha) \\ = \sum_\alpha (H^\alpha)^2\Big(\sum_{\beta\neq\alpha} L_\beta\Big) \geq 0. \tag{3.10}$$

And the equality implies $H^\alpha L_\beta = 0, \alpha \neq \beta$.

Lemma 3.1 (see [4], [8])

$$tr[(B_\alpha B_\beta - B_\beta B_\alpha)(B_\beta B_\alpha - B_\alpha B_\beta)] + \sum_{\alpha,\beta}[tr(B_\alpha B_\beta)]^2 \leq (2 - 1/p)\rho^4. \tag{3.11}$$

and equality implies one of the following (i) and (ii).

(i) $p = 1$,
(ii) *If* $p \geq 2$, *then* p *must be* 2. *After a suitable renumbering of the* e_{n+1}, e_{n+2}, *we have* $B_{n+1} = \lambda\tilde{A}, B_{n+2} = \mu\tilde{B}, \lambda\mu \neq 0$,

where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.12}$$

From (3.10) and (3.11) we get

$$Q \geq \rho^2(n - (2 - \frac{1}{p})\rho^2). \tag{3.13}$$

Lemma 3.2

Let M be an n-dimensional compact oriented Willmore submanifold in an $(n+p)$-dimensional unit sphere. Then

$$\int_M \left(\left(2 - \frac{1}{p}\right)\rho^2 - n \right) \rho^n dM \geq 0, \tag{3.14}$$

and equality implies $\nabla h = 0$ and one of the following (a) and (b):

(a) $p = 1$,
(b) $p \geq 2, \vec{H} = 0$.

Proof. Since the equality of (3.14) holds implies that equalities of (3.3), (3.10) and (3.11) hold, we see that $\nabla h = 0$ comes from Theorem 3.1, $\vec{H} = 0$ comes from (3.10) and (ii) of Lemma 3.1. This completes the proof of Lemma 3.2. □

Proof of the Main Theorem. If $p = 1$, then from Lemma 3.2, M is a hypersurface with parallel second fundamental form, and it is known that M is either total geodesic or $M = S^k(a) \times S^{n-k}(b), a^2 + b^2 = 1$. Since M satisfies Willmore condition (3.1), by using Theorem 3.1 in our paper [6] we have $a = \sqrt{\frac{n-k}{n}}$ and $b = \sqrt{\frac{k}{n}}$.

If $p \geq 2$, then from Lemma 3.2, we know M is minimal with $S = \rho^2 = n/(2-1/p)$. Following Chern, Do Carmo and Kobayashi' the result in [4], we know M is Veronese surface. This completes the proof of the main Theorem. □

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