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On the existence of k-normal curves of given degree and genus in projective spaces

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Abstract

We find some ranges for the 4-tuples of integers (d, g, n, r) for which there is a smooth connected non-degenerate curve of degree d and genus g, which is k-normal for every $k \leq r$.

1. Introduction

Our starting point was the following very natural question:

Question 1.1 *For what triples of integers (d, g, n) there is a smooth connected non-degenerate and linearly normal curve in $\mathbb{P}^n$ of degree d and genus g?*

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(Here $\mathbb{P}^n$ denotes the n-dimensional projective space over an algebraically closed field).

For $n = 3$ this question was raised in [13], Problem 4d.4.

Several papers were devoted to this question without the linear normality assumption, the so called Halphen existence problem (see, for example, [10], [12], [23], [5], [4], [20], [21], [18], [15]).

A complete answer to Question 1.1 is known for $n \leq 5$ (see [7] and [22]). For $n \geq 6$ some ranges are known for which the answer is positive (see for example [5]). Further ranges can be found easily from [4] and [21], as communicated to us by O. Păsărescu.

In this paper we deal with a more precise question, looking for the k-normal curves, where $k \geq 1$. We recall the definition.

DEFINITION 1.2. Let k be a positive integer and let $C \subset \mathbb{P}^n$ be a curve (i.e. a locally Cohen-Macaulay equidimensional subscheme of dimension 1). We will say that C is *k-normal* if the restriction map $\rho_{C,k} : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(C, \mathcal{O}_C(k))$ is surjective.

We will say that C is *strongly k-normal* if the restriction map

$$\rho_{C,j} : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \to H^0(C, \mathcal{O}_C(j))$$

is surjective for every $j \leq k$.

Clearly C is k-normal if and only if $H^1(C, \mathcal{I}_C(k)) = 0$ and is strongly k-normal if and only if $H^1(C, \mathcal{I}_C(j)) = 0$ for every $j \leq k$.

A 1-normal curve is also called *linearly normal.*

The precise question we address in this paper is:

Question 1.3 *Given a triple of integers (d, g, n) find an integer $r := r(d, g, n) \geq 1$ such that there is a smooth connected non-degenerate and strongly r-normal curve in $\mathbb{P}^n$ having degree d and genus g.*

We give answers in a number of cases, where r is easy to compute, and is also sharp with respect to curves lying on some particular kind of surfaces.

Our first result is the following Theorem, which holds in characteristic zero.

Theorem 1.4 *Fix integers d, g, n such that $n \geq 3$ and $0 \leq d - n < g < \frac{d^2}{4(n-1)} - \frac{n-1}{4}$.*

Set $r := \left\lfloor \frac{d - \sqrt{d^2 - 4(n-1)g}}{2(n-1)} \right\rfloor$, $d_0 := d - 2(n-1)r$, $g_0 := (n-1)r^2 - dr + g$. Then:

(i) *$r \geq 1$ and $0 \leq g_0 \leq d_0 - n$;*

(ii) *(characteristic zero) there is a strongly r-normal smooth connected non-degenerate curve $C \subset \mathbb{P}^n$ of degree d and genus g, lying on a $K3$ surface of degree $2(n-1)$ and satisfying:*
   (a) *$h^0(C, \mathcal{O}_C(j)) = (n-1)j^2 + 2$ for $1 \leq j \leq r$;*
   (b) *$h^1(C, \mathcal{I}_C(r+1)) = d_0 - n - g_0$.*

   In particular C is not strongly $(r+1)$-normal if and only if $g_0 < d_0 - n$.

Theorem 1.4 applies in particular when $n = 3$; for $n \geq 4$ we have the following Theorem which holds in arbitrary characteristic, and covers some different ranges for the triple (d, g, n).

Theorem 1.5 *Fix integers d, g, n such that $n \geq 4$ and $n - 1 \leq d - n < g$. Then:*

(i) *If $n = 4$, assume $g \leq \frac{d^2-2d-9}{12}$ and set $r := \left\lfloor \frac{d-1-\sqrt{(d-1)^2-12g}}{6} \right\rfloor$. Then $r \geq 1$, and there is a strongly r-normal (not strongly $(r+1)$-normal) smooth connected non-degenerate curve $C \subset \mathbb{P}^4$ of degree d and genus g lying on a Bordiga surface of degree 6 satisfying $h^0(C, \mathcal{O}_C(j)) = 3j^2 + j + 1$, for $1 \leq j \leq r$.*

(ii) *If $n = 5$, assume $g \leq \frac{d^2-3d-12}{14}$ and set $r := \left\lfloor \frac{2d-3-\sqrt{(2d-3)^2-56g}}{14} \right\rfloor$. Then $r \geq 1$, and there is a strongly r-normal (not strongly $(r+1)$-normal) smooth connected non-degenerate curve $C \subset \mathbb{P}^5$ of degree d and genus g lying on a Bordiga surface of degree 7 satisfying $h^0(C, \mathcal{O}_C(j)) = \frac{7}{2}j^2 + \frac{3}{2}j + 1$, for $1 \leq j \leq r$.*

(iii) *If $n \geq 6$ after setting $\delta := 3$ if $n = 6$ and $\delta := 4$ if $n \geq 7$, assume $g < \frac{(d-n)(d+n-\delta)}{4n-2\delta}$. Set $r := \left\lfloor \frac{d-\delta-\sqrt{(d-\delta)^2-8(2n-\delta)g}}{4n-2\delta} \right\rfloor$. Then $r \geq 1$, and there is a strongly r-normal (not strongly $(r+1)$-normal) smooth connected non-degenerate curve $C \subset \mathbb{P}^n$ of degree d and genus g lying on a smooth rational surface of degree $2n - \delta$ and satisfying $h^0(C, \mathcal{O}_C(j)) = \frac{2n-\delta}{2}j^2 + \frac{\delta}{2}j + 1$, for $1 \leq j \leq r$.*

We observe that the bound for g in item (iii) of Theorem 1.5 is weaker than the bound $g \leq \frac{(d-n)^2}{4n-2\delta}$ given in [5], hence Theorem 1.5 produces also a wider range for the Halphen existence problem.

2. The proofs

We begin with an easy but useful method to construct smooth strongly r-normal curves.

Remark 2.1. Recall that if $S \subseteq \mathbb{P}^n$ is an arithmetically Cohen-Macaulay (aCM for short) surface and H is its hyperplane divisor, then $H^1(\mathcal{O}_S(jH)) = H^2(\mathcal{I}_S(jH)) = 0$ for every $j \in \mathbb{Z}$. In particular $H^1(\mathcal{O}_S) = 0$, that is S is regular.

It follows also that if $C_0 \subset S$ is a curve, then for any $j \in \mathbb{Z}$ the linear system $|C_0 + jH|$ cuts out on C_0 a complete linear series, as one can see from the exact sequence:

$$\begin{aligned} 0 \quad &\to \quad H^0(\mathcal{O}_S(jH)) \to H^0(\mathcal{O}_S(C_0 + jH)) \to H^0(\mathcal{O}_{C_0}(jH + C_0)) \\ &\to \quad H^1(\mathcal{O}_S(jH)) = 0. \end{aligned}$$

In particular $h^0(\mathcal{O}_S(C_0 + jH)) = 0$ if and only if $h^0(\mathcal{O}_{C_0}(jH + C_0)) = 0$.

Proposition 2.2

Let $S \subseteq \mathbb{P}^n$ be a smooth aCM surface of degree s and sectional genus π, and let H be a hyperplane divisor of S. Let $C_0 \subseteq S$ be a reduced curve of degree d_0 and *arithmetic genus g_0 and denote by c the number of connected components of C_0. Let $r > 0$ be an integer and let $C \in |C_0 + rH|$. Then*

(i) *$d := \deg C = d_0 + rs$, and*
$p_a(C) = g_0 + rd_0 + \frac{sr(r-1)}{2} + r(\pi - 1)$, *or equivalently*
$p_a(C) = g_0 + (d - \frac{s}{2} + \pi - 1)r - \frac{s}{2}r^2$.

(ii) *C is strongly $(r-1)$-normal and $h^1(\mathcal{I}_C(r)) = c - 1$; in particular C is strongly r-normal if and only if C_0 is connected. Moreover C is $(r+1)$-normal if and only if C_0 is linearly normal.*

(iii) *A general $C \in |C_0 + rH|$ is smooth irreducible in any of the following cases:*
- (iiia) *$|C_0|$ is base-point free;*
- (iiib) *$K_S = 0$, C_0 is smooth irreducible and $g_0 > 0$;*
- (iiic) *$K_S = 0$, C_0 is smooth, $g_0 = 0$, $d_0 \geq 2$ and either $r \geq 2$ or the characteristic is zero;*
- (iiid) *$C_0 = L_1 + \cdots + L_c$, where $L_1, \ldots, L_c$ are pairwise disjoint straight lines with self-intersection -1, and either $r \geq 2$ or the characteristic is zero.*

Proof. (i) The computation of $\deg C$ is obvious. Moreover from the adjunction formula we get first $K_S \cdot H = 2\pi - 2 - s$ and then the expression for $p_a(C)$.

(ii) If C_0 is reduced we have $h^1(\mathcal{I}_{C_0}(j)) = 0$ for $j < 0$ and $h^1(\mathcal{I}_{C_0}) = c - 1$. We have also that C_0 is linearly normal if and only if $H^1(\mathcal{I}_{C_0}(1)) = 0$. Moreover by Gorenstein liaison (see [17], Corollary 5.3.4) we have $H^1(\mathcal{I}_{C_0}(j)) = H^1(\mathcal{I}_C(j + r))$ for every $j \in \mathbb{Z}$, whence the conclusion.

(iii) If we are in case (iiia) it is easy to see that $|C_0 + rH|$ is very ample and the conclusion follows.

Assume (iiib). Then the linear system $|C_0|$ cuts out on C_0 the complete canonical series (see Remark 2.1). Hence if $g_0 > 0$ it follows that $|C_0|$ has no base points on C_0, whence has no base points. Then the previous case applies.

For the remaining cases it is sufficient to show that $|C_0 + rH|$ is base-point free whenever $r \geq 1$. Indeed if $r = 1$ one can use Bertini and if $r > 1$ case (iiia) applies with r replaced by $r - 1$ and C_0 replaced by a general curve of $|C_0 + rH|$.

Assume (iiic). By Remark 2.1 the linear system $|C_0 + rH|$ cuts out on C_0 a complete linear series. Since $C_0^2 = -2$ by adjunction, this linear series has degree $rd_0 + C_0^2 \geq d_0 - 2 \geq 0$ and hence it is base-point free, being C_0 rational. This easily implies that $|C_0 + rH|$ is base-point free, as claimed.

Now assume (iiid). Let $1 \leq t \leq c$ and set $Y_t := E_1 + \cdots + E_t$. Observe that $(Y_t + rH) \cdot E_j = r - 1 \geq 0$ for every $j = 1, \ldots, t$ and $(Y_t + rH) \cdot E_j = r > 0$ for $j = t + 1, \ldots, c$. Since E_j is rational it follows that $H^1(\mathcal{O}_{E_j}(C_0 + rH)) = 0$ for every j, whence the exact sequence

$$\cdots \to H^1(\mathcal{O}_S(Y_t + rH - E_j)) \to H^1(\mathcal{O}_S(Y_t + rH)) \to 0$$

and since S is aCM it follows easily by induction on t that $H^1(\mathcal{O}_S(Y_t + rH)) = 0$.

By the above observation it follows that the linear system $|C_0 + rH|$ cuts out on every E_j ($j = 1, \ldots c$) a complete linear series of non-negative degree, whence $|C_0 + rH|$ is base-point free (e.g. see the argument for the case (iiib)). $\square$

Lemma 2.3 below allows to compute the postulation of the linear series cut out on a k-normal curve lying on a aCM surface S by the linear system $|kH|$, where H is a hyperplane divisor on S. This Lemma will be needed in the proofs of both Theorem 1.4 and Theorem 1.5.

Lemma 2.4 shows that some $K3$ smooth surfaces are aCM, and this allows to apply our Proposition 2.2, in order to prove Theorem 1.4.

Lemma 2.3

Let $S \subseteq \mathbb{P}^n$ be an aCM surface of degree s, sectional genus π and let H be a hyperplane divisor of S. Let $C \subseteq S$ be a k-normal curve and assume that the linear systems $|kH - C|$ and $|K_S - kH|$ are non-effective. Then $h^0(\mathcal{O}_C(k)) = \frac{s}{2}k^2 + (\frac{s}{2} - \pi + 1)k + p_a(S) + 1$.

Proof. Since S is aCM we have $H^1(\mathcal{O}_S(C - kH)) = H^1(\mathcal{I}_C(k)) = 0$. Hence from the exact sequence $0 \to \mathcal{O}_S(kH - C) \to \mathcal{O}_S(kH) \to \mathcal{O}_C(kH) \to 0$ we get $h^0(\mathcal{O}_C(k)) = h^0(\mathcal{O}_S(k))$.

Now we have $H^1(\mathcal{O}_S(k)) = H^2(\mathcal{I}_S(k)) = 0$ since S is aCM and $H^2(\mathcal{O}_S(k)) = H^0(\mathcal{O}_S(K_S - kH)) = 0$ by assumption. Then $h^0(\mathcal{O}_C(k)) = \chi(\mathcal{O}_S(k))$ and the conclusion follows easily by using the hyperplane sequence. □

Lemma 2.4

Any smooth $K3$ surface $S \subset \mathbb{P}^n$ $(n \geq 3)$ of degree $2n - 2$ is non-degenerate, aCM and has sectional genus n.

Proof. Let $H = S \cap L$ be a general hyperplane section of S. Since $K_S = 0$ and $H^1(\mathcal{O}_S) = 0$ the linear system $|L|$ cuts out on H the complete canonical series, whence the embedding $H \subset L$ is the canonical model of H. In particular $H \subset L$ is non-degenerate and projectively normal (see e.g. [1]), whence S is non-degenerate and aCM. The last assertion is obvious. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. (i) Consider the real polynomial $g(x) := (n-1)x^2 - dx + g$ and let Δ be its discriminant.

Since $\Delta > (n-1)^2$ and $g(1) \geq 0$, g has two real roots x_1 and x_2 with $1 \leq x_1 < x_2 - 1$. By assumption $r := \lfloor x_1 \rfloor$, whence $r \geq 1$, $g(r) \geq 0$ and $g(r+1) < 0$. The conclusion follows since $g_0 = g(r)$ and $g(r+1) = g_0 - d_0 + n - 1$.

(ii) An easy calculation shows that $g_0 \leq d_0 - n$ implies $g_0 < \frac{d_0^2}{4(n-1)}$ and $(d_0, g_0) \neq (2n-1, n)$. Hence by [15], Theorem 4.6 (see also [18] for $n = 3$) there are a smooth $K3$ surface $S \subseteq \mathbb{P}^n$ of degree $2(n-1)$ and a smooth connected curve $C_0 \subseteq S$ of genus g_0 and degree d_0 such that $\mathrm{Pic}(S)$ is freely generated by the classes of C_0 and H, where H is a hyperplane divisor (here characteristic zero is needed).

The surface S is non-degenerate, has sectional genus n and is aCM by Lemma 2.4. Let C be a general curve in the linear system $|C_0 + rH|$. Then C is smooth, connected, of degree d and genus g and strongly r-normal by Proposition 2.2 (i), (ii), (iiib), (iiic), and satisfies (a) by Lemma 2.3.

In order to prove (b) we use first the above mentioned structure of $\mathrm{Pic}(S)$ to show that $H^0(\mathcal{O}_S(C_0 - H)) = H^0(\mathcal{O}_S(H - C_0)) = 0$ (see [3], proof of Proposition 5.7, with obvious changes). In particular C_0 is non-degenerate. Moreover by Remark 2.1 we have $h^0(\mathcal{O}_{C_0}(C_0 - H)) = 0$, whence C_0 is non-special, since $\omega_{C_0} = \mathcal{O}_{C_0}(C_0)$. Then $h^0(\mathcal{O}_C(1)) = d_0 - g_0 + 1$, and the conclusion follows easily. □

Now we need some prepapatory results, in order to prove Theorem 1.5.

Lemma 2.5

Let $S \subset \mathbb{P}^n$ $(n \geq 4)$ be a smooth, regular, non-degenerate, linearly normal surface of degree s and sectional genus π. Assume $s \geq 2\pi + 1$. Then S is aCM.

Proof. Let $H := S \cap L$ be a general hyperplane section. Since $h^1(\mathcal{O}_S) = 0$, $H \subset L$ is a linearly normal curve of degree $s \geq 2\pi + 1 = 2p_a(H) + 1$. By [2] or [16] or [19] (see the introduction of [9]) or [9], Theorem 1, the embedding of H in L is projectively normal. Hence S is aCM. □

Corollary 2.6

Let $S \subset \mathbb{P}^n$ $(4 \leq n \leq 14)$ be a Bordiga surface, i.e. a smooth surface isomorphic to the blowing-up of $\mathbb{P}^2$ at $14 - n$ general points, and embedded by the complete linear system of all quartics through them. Then S is aCM, has degree $n + 2$ and sectional genus 3.

Proof. S has degree $n+2$ and sectional genus 3 by construction. If $n \geq 5$ the conclusion follows immediately from Lemma 2.5. Let now $n = 4$. Even this case is classically well-known, but we prefer to give a proof. Let $H = S \cap L$ be a general hyperplane section of S. Hence H is isomorphic to a smooth plane quartic curve. Thus H has genus 3 and it is not hyperelliptic because $\omega_H \cong \mathcal{O}_H(1)$ is very ample. Since $h^1(\mathcal{O}_S) = 0$, the embedding of H in L is linearly normal. By [14] (or, in arbitrary characteristic, [9], Theorem 1 and 0.1) $H \subset L$ is a projectively normal embedding. Hence S is aCM also in this case. □

Corollary 2.7

Let $S \subset \mathbb{P}^n$ $(n \geq 4)$ be a non-degenerate, smooth, linearly normal, rational surface of degree $2n - \delta$ $(\delta \geq 3)$. Then S is aCM, and has sectional genus $n - \delta + 1$.

Proof. Let $H = S \cap L$ be a general hyperplane section of S. Then $H \subset L$ is a non-degenerate linearly normal curve of degree $2n - \delta$, which is non-special by Clifford. Hence its genus is $n - \delta + 1$ by Riemann-Roch. The conclusion follows by Lemma 2.5. □

Now we can prove Theorem 1.5.

Proof of Theorem 1.5. (i) Consider the real polynomial $g(x) := 3x^2 - (d-1)x + g$ and let Δ be its discriminant.

Since $\Delta > 9$ and $g(1) \geq 0$, $g(x)$ has two real roots x_1 and x_2 with $1 \leq x_1 < x_2 - 1$. By assumption $r = \lfloor x_1 \rfloor$, whence $r \geq 1$, $g(r) \geq 0$ and $g(r+1) < 0$. Then, if we set

$g_0 := g(r)$ and $d_0 := d - 6r$, we have $0 \leq g_0 \leq d_0 - 5$. Hence by [23], Theorem 1.2.2, there is a smooth connected curve C_0 of degree d_0 and genus g_0 lying on a Bordiga surface S of degree 6, clearly not linearly normal. Moreover from the proof it follows also that $|C_0|$ is base-point free. Let C be a general curve in the linear system $|C_0+rH|$, where H is a hyperplane divisor. Then C satisfies the requirements by Corollary 2.6, Proposition 2.2 (i), (ii), (iiia), and Lemma 2.3.

(ii) Consider the real polynomial $g(x) := \frac{7}{2}x^2 - \frac{2d-3}{2}x + g$.

The same proof as in (i) works using the degree 7 Bordiga surface, just quoting [23], Proposition 2.2.1.

(iii) Consider the real polynomial $g(x) := \frac{1}{2}(2n - \delta)x^2 - \frac{2d-\delta}{2}x + g$.

The proof is the same as before, using [5], Proposition 5. Indeed by this proposition there is a surface S satisfying the assumptions of Corollary 2.7 (hence aCM) and for any pair of integers d_0 and g_0 such that $0 \leq g_0 \leq d_0 - n - 1$ there is a smooth connected curve $C_0 \subset S$ having degree d_0 and genus g_0. Moreover from the proof it follows also that $|C_0|$ is base-point free. □

3. Further examples and remarks

The method used to prove Theorems 1.4 and 1.5 is based on the general principle established in Proposition 2.2, namely to construct the desired curves starting from known ones lying on suitable aCM surfaces.

In this way we could answer our Question 1.3, finding some ranges for the 4-tuples (d, g, n, r) for which there are smooth connected non-degenerate curves of degree d and genus g in $\mathbb{P}^n$ which are, in addition, strongly r-normal.

As we have seen from Proposition 2.2, the first problem is arithmetic, and in our cases it was an easy one.

Wider ranges could be found, at least in principle, by using curves lying on other surfaces, as for example the cubic surface in $\mathbb{P}^3$ (see [10] and [12]), the Del Pezzo surfaces in $\mathbb{P}^4$, $\mathbb{P}^5$ (see [23]), and other rational surfaces in $\mathbb{P}^n$ ($n \geq 6$) (see [4], [20]). But it is immediate to see that here the arithmetical part becomes much more complicated, hence it seems much harder to give clean statements for wider ranges.

It is however easy to produce curves not falling in the ranges of Theorems 1.4 and 1.5, just starting from disjoint union of lines. Here the arithmetic part is irrelevant.

We give some examples below, as a straightforward application of Proposition 2.2.

EXAMPLE 3.1: Let $3 \leq n \leq 8$ and let $S \subseteq \mathbb{P}^n$ be a Del Pezzo surface. Then S is rational and by construction it has degree n and sectional genus 1. Hence S is aCM by Lemma 2.5. Moreover it contains $9 - n$ pairwise disjoint lines with self-intersection -1. Fix integers c and t such that $t \geq 1$ and $1 \leq c \leq 9 - n$ and set:

$$d := c + n(t+1)$$
$$g := 1 + ct + \frac{nt(t+1)}{2}$$

Then by Proposition 2.2 S contains a smooth connected non-degenerate curve C of degree d and genus g, which is aCM if $c = 1$ and strongly t (and not $(t+1)$)-normal if $c \geq 2$.

EXAMPLE 3.2: Let $n \geq 6$ and write $n = 3k + h$ $(0 \leq h \leq 2)$. Let $S \subseteq \mathbb{P}^n$ be the rational surface constructed in [4]. S has degree $n - k + 1$, sectional genus k and contains $5 - h$ pairwise disjoint lines with self intersection -1. Then by Lemma 2.5 S is aCM. Fix integers c and t such that $t \geq 1$ and $1 \leq c \leq 5 - h$ and set:

$$d := c + (n - k + 1)(t + 1)$$
$$g := 1 + ct + \frac{(n-k+1)t(t+1)}{2} + k - 1$$

Then by Proposition 2.2 S contains a smooth connected non-degenerate curve of degree d and genus g, which is aCM if $c = 1$ and strongly t (and not $(t+1)$)-normal if $c \geq 2$.

Remark 3.3. If S is either a smooth cubic surface in $\mathbb{P}^3$ or a smooth surface of minimal degree in any projective space and $C \subset S$ is any curve, the first cohomology of C is known (see [8] and [6]). Hence a complete classification of the strongly r-normal curves lying on S is in principle possible. Moreover from the above mentioned papers it follows also that if C is r-normal and $r < \max\{j \in \mathbb{Z} \mid H^1(\mathcal{I}_C(j)) \neq 0\}$, then C is strongly r-normal.

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