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Involutions in mapping class groups of non-orientable surfaces

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Abstract

We prove that the mapping class group and the pure mapping class group of closed non-orientable surface with punctures are generated by involutions.

1. Introduction

Let N be a closed *non-orientable* surface of genus g with n punctures. The mapping class group $\mathcal{M}_{g,n}$ is the group of isotopy classes of all homeomorphisms of N which preserve the set of punctures. Denote by $\mathcal{PM}_{g,n}$ the subgroup of $\mathcal{M}_{g,n}$ consisting of the isotopy classes of those homeomorphisms of N which fix each puncture. The group $\mathcal{PM}_{g,n}$ is called the pure mapping class group. When $n = 0$ (no punctures) we will denote the mapping class group simply by $\mathcal{M}_g$.

In [5] and [6] Lickorish considered the problem of determining generators for $\mathcal{M}_g$ and, using Lickorish's partial results, Chillingworth [3] determined a finite set of generators for that group. Later Birman and Chillingworth [1] showed how to obtain generators for the mapping class group of a non-orientable surface from generators of the mapping class group of its orientable double cover. We used the generators given in [1] to prove that the mapping class group of closed non-orientable surface is generated by involutions. Later we discovered that the manuscript [1] contained a mistake, i.e. the set of elements given in Theorem 2 of [1] does not generate $\mathcal{M}_g$ in the case $g = 4$; one more generator is needed in that case (see [2]). The main result of [7] is true also in the case $g = 4$ and the proof works without significant changes for the corrected set of generators.

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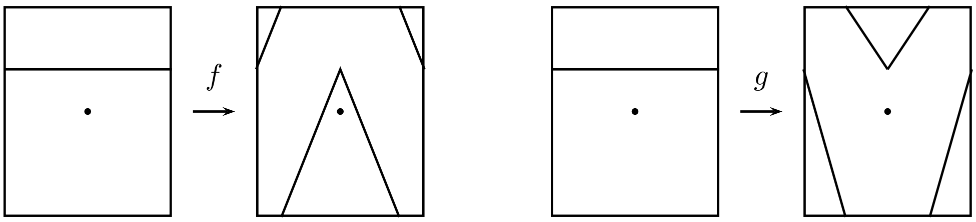

Figure 1. The puncture slide.

In this paper we prove that $\mathcal{M}_{g,n}$ and $\mathcal{PM}_{g,n}$ are generated by involutions. We use the generators of the mapping class groups given by Chillingworth [3] and Korkmaz [4]. In the case $n = 0$ this gives a new proof of our result [7] without referring to [1].

The paper is organized as follows. In Section 2 we recall definitions of several homeomorphisms of the Möbius strip and the holed Klein bottle, and show how to express each of those homeomorphisms as a product of two involutions. In Section 3 we prove that $\mathcal{M}_{g,n}$ and $\mathcal{PM}_{g,n}$ are generated by involutions.

2. Homeomorphisms of non-orientable surfaces

The most important (from the point of view of the structure of the mapping class group) surface homeomorphism is the well known Dehn twist. It is a homeomorphism associated with a simple closed two-sided curve with oriented tubular neighborhood, supported in a neighborhood of the curve and depending up to isotopy only on the isotopy class of the curve. For a definition of Dehn twist see for example [5]. In this section we will recall definitions of several homeomorphism which are specific for non-orientable surfaces as they are associated with one-sided curves.

2.1. Puncture slide

We will now describe the self-homeomorphism of the Möbius strip with a puncture called *puncture slide*. This homeomorphism is the identity on the boundary and we will later extend it to obtain a homeomorphism of a closed non-orientable surface with punctures. We find it convenient to give a more rigorous definition than it is done in other papers (see for example [4]).

Let us consider the rectangle $[-1, 1] \times [0, 1]$ with puncture $(0, \frac{1}{2})$ divided out by the relation $(x, 0) \sim (-x, 1)$ as a model for the Möbius strip. Denote this model by M. Define the puncture slide f by the formula:

$$f([x, y]) = \begin{cases} [x, y + 1 - |x|] & \text{for} \quad y \leq |x|, \\ [-x, y - |x|] & \text{for} \quad y \geq |x|, \end{cases} \tag{1}$$

where $[x, y]$ denotes the class of a point (x, y) in M. It is easy to check that the above formula defines a homeomorphism of the punctured Möbius strip.

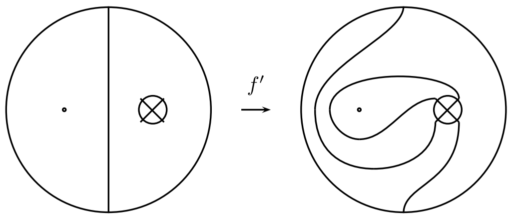

Figure 2. The puncture slide (again).

Let S denote the symmetry $S([x,y]) = [x, 1-y]$ and consider the conjugation $g = SfS$. We have

$$g([x,y]) = \begin{cases} [x, y+|x|-1] & \text{for} \quad y \geq 1-|x|, \\ [-x, y+|x|] & \text{for} \quad y \leq 1-|x|, \end{cases} \tag{2}$$

end it is easy to check that $fg = gf = \mathrm{id}$, so $g = f^{-1}$. The homeomorphism f^{-1} is the slide in the opposite direction. Figure 1 shows how f and g act on an interval joining two points on the boundary of M.

We can also use another model M' of the Möbius strip: remove from a disk an interior of a smaller disk and identify antipodal points on the boundary of the smaller disk. Choose a homeomorphism h from M to M' such that the composition $S' = hSh^{-1}$ is the homeomorphism induced by the reflection of the disk about the line containing the puncture and the center of the removed disk. Setting $f' = hfh^{-1}$ we have $S'f'S' = f'^{-1}$. Figure 2 is the analogue of Figure 1 and it shows the action of the puncture slide f' on M'.

Let N be a non-orientable surface with at least one puncture. There is an embedding $e\colon M' \to N$. We can extend $ef'e^{-1}$ to a homeomorphism ϕ_e of N by the identity outside $e(M')$. We will call puncture slide any homeomorphism of N isotopic to some ϕ_e.

Note that if we can also extend $eS'e^{-1}$ to an involution σ of N, then we have $\sigma\phi_e\sigma = \phi_e^{-1}$ and ϕ_e is a product of homeomorphisms of order 2: $\phi_e = (\sigma)(\sigma\phi_e)$.

2.2. Crosscap slide

Now we describe a homeomorphism of the Klein bottle with a hole called *crosscap slide*. The Klein bottle with a hole is homeomorphic to the Möbius strip from which an interior of a small disk has been removed and antipodal points on the boundary of the disk have been identified. Crosscap slide is an analogue of puncture slide, except that instead of puncture, we now slide a crosscap along the core of the Möbius strip. Just as in the case of puncture side we can observe the effect of crosscap slides y and y' on intervals in two different models of the Klein bottle with a hole K and K' (Figure 3).

We have the identities: $SyS = y^{-1}$ and $S'y'S' = y'^{-1}$ where S and S' are involutions of K and K': S is induced by the symmetry $(x,y) \to (x, 1-y)$ of the rectangle and S' is induced by the reflection of the disk about the line containing the centers of the removed disks.

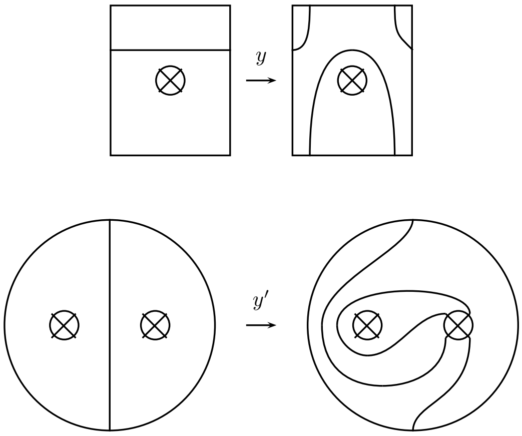

Figure 3. The crosscap slide.

If $e\colon K' \to N$ is an embedding of the Klein bottle with a hole into a non-orientable surface of genus at least 2, then $ey'e^{-1}$ extends to ψ_e by the identity outside $e(K')$. Any homeomorphism of a non-orientable surface isotopic to some ψ_e will be called a crosscap slide. Again, if $eS'e^{-1}$ extends to an involution σ of N, then we have $\sigma\psi_e\sigma = \psi_e^{-1}$ and we can write ψ_e as a product of homeomorphisms of order 2: $\psi_e = (\sigma)(\sigma\psi_e)$.

The homeomorphism ψ_e^2 is isotopic to a Dehn twist along the boundary of $e(K')$. Crosscap slide is also called y-homeomorphism and was introduced by Lickorish [5].

3. Generators of the mapping class groups

Suppose that N is a non-orientable surface with boundary and let c be a boundary component homeomorphic to a circle. We can slide c along a core of a Möbius strip to obtain a homeomorphism we will call *boundary slide*. Note that such defined homeomorphism reverses orientation on c.

Suppose now that N is a non-orientable surface with punctures and $c \subset N$ is a nonseparating two-sided simple closed curve, such that the surface N_c obtained from N by cutting along c is non-orientable. Then N_c is a connected non-orientable surface having two more boundary components than N. The additivity of the Euler characteristic implies that N_c has the same Euler characteristic as N. Hence, by the classification of surfaces, N_c is determined up to a homeomorphism by N and the fact that c is nonseparating with N_c non-orientable.

Lemma 1

If c and c' are two nonseparating two-sided simple closed curves on N such that the surfaces N_c and $N_{c'}$ are non-orientable, then there is a homeomorphism $f\colon N \to N$ such that $f(c) = c'$.

Proof. By the remarks preceding the Lemma, N_c and $N_{c'}$ are homeomorphic. Let $F\colon N_c \to N_{c'}$ be a homeomorphism. Let c_1 and c_2 be two boundary components

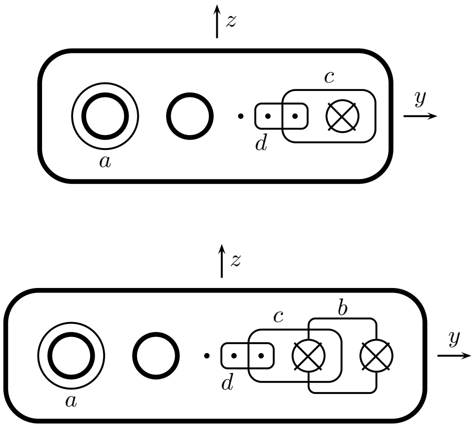

Figure 4. $g = 5, 6, n = 3$.

of N_c resulting from c and let $c'_1 = F(c_1)$ and $c'_2 = F(c_2)$. Let $j\colon c_1 \to c_2$ and $j'\colon c'_1 \to c'_2$ be two gluing homeomorphisms giving N back from N_c and $N_{c'}$. If F agrees with the gluings, i.e. if $F \circ j = j' \circ F$ on c_1 then F gives rise (by gluing) to a homeomorphism $f\colon N \to N$ such that $f(c) = c'$. Composing F with a boundary slide if necessary we can assume that $F \circ j, j' \circ F\colon c_1 \to c'_2$ are both orientation-preserving (or both orientation-reversing). Since every two orientation-preserving (as also every two orientation-reversing) homeomorphism of a circle are isotopic, we can find a homeomorphism $F'\colon c_2 \to c'_2$ isotopic to $F|_{c_2}$ and such that $F' \circ j = j \circ F|_{c_1}$. By extending the isotopy between $F|_{c_2}$ and F' to an isotopy of F (and without changing $F|_{c_1}$) we get a homeomorphism $F\colon N_c \to N_{c'}$ which agrees with gluings. This completes the proof. □

Let N be a closed non-orientable surface of genus g with n punctures $z_1, \ldots, z_n$. The surface N can be obtained from an orientable surface O with n punctures by removing interiors of one or two disks (depending on weather g is odd or even) and identifying antipodal points on the boundary. (Sometimes we will use another model for N: a sphere with n punctures and g crosscaps.) Suppose that the surface O is embedded in $\mathbb{R}^3$ in such a way that the reflection in the xy plane induces an involution σ of N (Figure 4).

Suppose that tubular neighborhoods of the two-sided curves a and b are oriented. Denote by t_a and t_b the isotopy classes of Dehn twists about these curves regarded as homeomorphisms of N and by s the isotopy class of σ. Then in the mapping class group $\mathcal{M}_{g,n}$ we have: $st_as = t_a^{-1}$ and $st_bs = t_b^{-1}$ and so we can write t_a and t_b as products of involutions: $t_a = s(st_a)$, $t_b = s(st_b)$.

Suppose that c is any two-sided nonseparating simple closed curve with oriented tubular neighborhood such that N_c is non-orientable. By Lemma 1 there is a homeomorphism h of N such that $h(c) = a$. Then $t_c = [h]^{-1}t_a^{\epsilon}[h]$ where $[h]$ denotes the isotopy class of h in the mapping class group and ϵ is 1 if h preserves the orientations

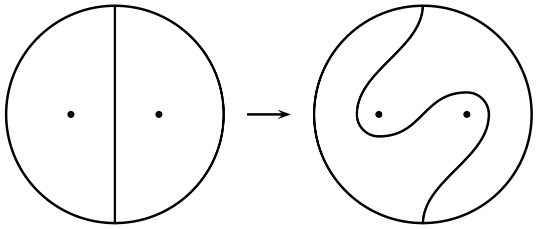

Figure 5. The elementary braid.

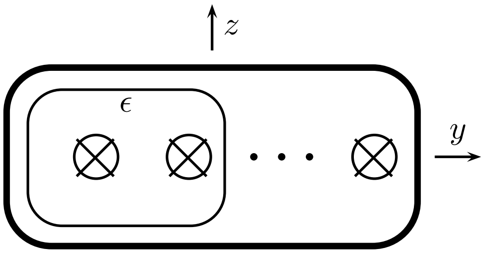

Figure 6.

of tubular neighborhoods and -1 otherwise. Since t_a and t_a^{-1} are conjugate in $\mathcal{M}_{g,n}$, any two isotopy classes of Dehn twists about two-sided nonseparating simple closed curves whose complements are non-orientable are conjugate in $\mathcal{M}_{g,n}$ and each t_c can be expressed as product of two involutions.

There is an embedding $e\colon M' \to N$ of punctured Möbius strip into N such that c is the boundary of $e(M')$ and $\sigma|_{e(M')} = eS'e^{-1}$. Denote by v the isotopy class of the puncture slide ϕ_e. It follows from Section 2.1 that $svs = v^{-1}$ and v is also a product of involutions.

Also the isotopy class $t_d^{1/2}$ of an elementary braid interchanging the two points inside d (Figure 5) is a product of two involutions since $st_d^{1/2}s = (t_d^{1/2})^{-1}$.

Let N' be the sphere with n punctures and g crosscaps in Figure 6 and let $h\colon N' \to N$ be any homeomorphism. Suppose that $g \geq 2$. Then there is an embedding $e\colon K' \to N'$ such that ϵ is the boundary of $e(K')$ and $eS'e^{-1}$ extends to the involution σ' of N' induced by the reflection in the xy plane. Then it follows from Section 2.2 that $\sigma'\psi_e\sigma' = \psi_e^{-1}$. Denoting by y and s' the isotopy classes of $\psi_{he} = h\psi_e h^{-1}$ and $h\sigma' h^{-1}$ respectively, we obtain another element of $\mathcal{M}_{g,n}$ that is a product of two involutions: $y = s'(s'y)$. Now ψ_{he} is a crosscap slide such that ψ_{he}^2 is Dehn twist along the curve $h(\epsilon)$. Note that ϵ is a separating curve with both components of $N' \backslash \epsilon$ non-orientable if $g \geq 3$ and so $h(\epsilon)$ has the same property. The next theorem is composed of Theorems 4.5, 4.11 and 4.15 of [4] (in the case when $n = 0$ see also [3] and [5]).

Theorem 2

If $g \geq 3$ and $g \neq 4$ then the mapping class group $\mathcal{M}_{g,n}$ is normally generated by $\{t_a, y\}$ if $n = 0$, $\{t_a, y, v\}$ if $n = 1$ and $\{t_a, y, v, t_d^{1/2}\}$ if $n \geq 2$. The group $\mathcal{M}_{4,n}$ is generated by the above set together with t_b.

The group $\mathcal{M}_{2,n}$ is normally generated by $\{t_b, y\}$ if $n = 0$, $\{t_b, y, v\}$ if $n = 1$ and $\{t_b, y, v, t_d^{1/2}\}$ if $n \geq 2$.

The group $\mathcal{M}_{1,n}$ is trivial if $n = 0$, normally generated by $\{v\}$ if $n = 1$ and $\{v, t_d^{1/2}\}$ if $n \geq 2$.

As a corollary we obtain:

Theorem 3

The mapping class group $\mathcal{M}_{g,n}$ is generated by involutions.

Proof. Denote by G the subgroup of $\mathcal{M}_{g,n}$ generated by involutions. Since each of the elements $t_a, t_b, y, v, t_d^{1/2}$ is a product of involutions, G contains the subgroup H generated by these elements. Since G is normal in $\mathcal{M}_{g,n}$, it contains the normal closure of H, which is $\mathcal{M}_{g,n}$ by Theorem 2. □

The group $\mathcal{PM}_{g,n}$ is obviously normal in $\mathcal{M}_{g,n}$ and we have a short exact sequence induced by the action of $\mathcal{M}_{g,n}$ on punctures:

$$1 \to \mathcal{PM}_{g,n} \to \mathcal{M}_{g,n} \to \Sigma_n \to 1, \quad (3)$$

where Σ_n denotes the symmetric group on n letters. Note that t_a, t_b, y and v are in $\mathcal{PM}_{g,n}$.

The next theorem follows from Theorems 4.3, 4.9 and 4.13 of [4]:

Theorem 4

Suppose that $n \geq 1$ and let H denote the subgroup of $\mathcal{PM}_{g,n}$ generated by $\{t_a, t_b, y, v\}$ if $g \geq 3$, $\{t_b, y, v\}$ if $g = 2$ and $\{v\}$ if $g = 1$. Then $\mathcal{PM}_{g,n}$ is the normal closure of H in $\mathcal{M}_{g,n}$.

We obtain the following theorem.

Theorem 5

The pure mapping class group $\mathcal{M}_{g,n}$ is generated by involutions.

Proof. Denote by G the subgroup of $\mathcal{PM}_{g,n}$ generated by involutions. Note, that since s and s' are both in $\mathcal{PM}_{g,n}$ we have that t_a, t_b, y and v are in G. The group G is characteristic in $\mathcal{PM}_{g,n}$, hence normal in $\mathcal{M}_{g,n}$. It follows that G contains the normal closure of H in $\mathcal{M}_{g,n}$, which is $\mathcal{PM}_{g,n}$ by Theorem 4. □

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