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The support theorem for the complex Radon transform of distributions

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Abstract

The complex Radon transform $\hat{F}$ of a rapidly decreasing distribution $F \in \mathcal{O}'_C(\mathbb{C}^n)$ is considered. A compact set $K \subset \mathbb{C}^n$ is called linearly convex if the set $\mathbb{C}^n \setminus K$ is a union of complex hyperplanes. Let $\hat{K}$ denote the set of complex hyperplanes which meet K. The main result of the paper establishes the conditions on a linearly convex compact K under which the support theorem for the complex Radon transform is true: from the relation $\operatorname{supp}(\hat{F}) \subset \hat{K}$ it follows that $F \in \mathcal{O}'_C(\mathbb{C}^n)$ is compactly supported and $\operatorname{supp}(F) \subset K$.

If f is the function defined on $\mathbb{R}^n$ ($\mathbb{C}^n$), the classical real (complex) Radon transform Rf of f is the function defined on hyperplanes; the value of Rf at a given hyperplane is the integral of f over that hyperplane. For the theory of the Radon transform we refer to J. Radon [11], F. John [6], [7], I.M.Gel'fand, M.I.Graev, and N.Ya. Vilenkin [1], S. Helgason [2], [3], D. Ludwig [8], A. Hertle [4]. One of the basic results on the classical Radon transform is Helgason's support theorem [2]: A rapidly decreasing function must vanish outside a ball if its real Radon transform does. This theorem holds for every convex compact set in $\mathbb{R}^n$ and remains valid for rapidly decreasing distributions [4].

In the present paper we prove the support theorem for the complex Radon transform of distributions.

Notations. For $z, w \in \mathbb{C}^n$ we write $\langle z, w\rangle = \sum z_j w_j . B^n(z,R) := \{w \in \mathbb{C}^n \mid |w - z| < R\}$ denotes the euclidean ball of center z and radius r in $\mathbb{C}^n$. If X is a set, we

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denote by $\bar{X}$ the closure of X. The standard Lebesgue measure in $\mathbb{C}^n$ is $d\omega_{2n}$. S^{2n-1} denotes the unit sphere in $\mathbb{C}^n$, and $d\sigma$ is the area element on S^{2n-1}. For n-tuples $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ of non-negative integers, we denote by $\partial^p \bar{\partial}^q$ the partial derivative

$$\frac{\partial^{|p|+|q|}}{\partial z_1^{p_1} \ldots \partial z_n^{p_n} \partial \bar{z}_1^{q_1} \ldots \partial \bar{z}_n^{q_n}}$$

of order $|p| + |q| = p_1 + \ldots + p_n + q_1 + \ldots + q_n$. Similarly, for $z = (z_1, \ldots, z_n)$ we write $z^p = z_1^{p_1} \ldots z_n^{p_n}$, $\bar{z}^q = \bar{z}_1^{p_1} \ldots \bar{z}_n^{q_n}$. For a domain $\Omega \subset \mathbb{C}^n$, we denote by $\mathcal{S}(\Omega)$, $\mathcal{D}(\Omega)$, and $\mathcal{E}(\Omega)$ the spaces of rapidly decreasing C^∞ functions, C^∞ functions with compact support, and C^∞ functions, respectively. The dual spaces $\mathcal{S}'(\Omega)$, $\mathcal{D}'(\Omega)$, and $\mathcal{E}'(\Omega)$ are the spaces of tempered distributions, distributions, and distributions with compact support, respectively.

If $\varphi \in \mathcal{S}(\mathbb{C}^n)$, the standard complex Radon transform of φ (denoted by $\hat{\varphi}$) is defined by

$$\hat{\varphi}(\xi, s) = \frac{1}{|\xi|^2} \int\limits_{\langle z, \xi \rangle = s} \varphi(z)\, d\lambda(z), \tag{1}$$

where $(\xi, s) \in (\mathbb{C}^n \setminus 0) \times \mathbb{C}$, and $d\lambda(z)$ is the area element on the hyperplane $\{z : \langle z, \xi \rangle = s\}$. For a set $A \subset \mathbb{C}^n$, we denote by $\hat{A}$ the set of all $(\xi, s) \in (\mathbb{C}^n \setminus 0) \times \mathbb{C}$ such that the hyperplane $\{z : \langle z, \xi \rangle = s\}$ meets A. A set $A \subset \mathbb{C}^n$ is called linearly convex if, for every $w \notin A$, there is a complex hyperplane $\{z : \langle z, \xi \rangle = s\}$ which contains w and does not meet A (see Martineau [9]).

We use the approach of Gel'fand et al. [1] to introduce the complex Radon transform of distributions. Let $X = S^{2n-1} \times \mathbb{C}$, and let $\mathcal{E}(X)$ be the set of complex-valued functions $\varphi(w, s)$ on $S^{2n-1} \times \mathbb{C}$ which satisfy the following conditions:

(a) Functions $\varphi(w, s)$ are infinitely differentiable with respect to s.

(b) For all $p, q \geq 0$ the derivatives

$$\frac{\partial^{p+q} \varphi(w, s)}{\partial s^p \partial \bar{s}^q}$$

are continuous on $S^{2n-1} \times \mathbb{C}$.

(c) $\varphi(we^{i\theta}, se^{i\theta}) = \varphi(w, s)$ for all $\theta \in [0, 2\pi]$.

We give $\mathcal{E}(X)$ the topology defined by the system of seminorms

$$q_k(f) = \max_{k_1 + k_2 \leq k} \max_{|s| \leq k} \max_{w \in S^{2n-1}} \left| \frac{\partial^{k_1 + k_2} f(w, s)}{\partial s^{k_1} \partial \bar{s}^{k_2}} \right|.$$

By $\mathcal{D}(X)$ we denote the space of all compactly supported functions in $\mathcal{E}(X)$. We give $\mathcal{D}(X)$ the standard topology of the inductive limit of the spaces

$$\mathcal{D}_m = \left\{ \varphi \in \mathcal{E}(X) : \operatorname{supp}(\varphi) \subset S^{2n-1} \times \{|s| \leq m\} \right\}.$$

Let $R\mathcal{D}(X)$ be the subspace of $\mathcal{D}(X)$ formed by the Radon transforms $\hat{\varphi}$ of functions in $\mathcal{D}(\mathbb{C}^n)$ (the equality $\hat{\varphi}(we^{i\theta}, se^{i\theta}) \equiv \hat{\varphi}(we^{i\theta}, se^{i\theta})$ follows for $\varphi \in \mathcal{D}(\mathbb{C}^n)$ from the definition of $\hat{\varphi}$). Similarly, we define the subspace $R\mathcal{S}(X)$ of $\mathcal{S}(X)$.

The dual Radon transform is the operator $R^* : \mathcal{E}(X) \to \mathcal{E}(\mathbb{C}^n)$ given by

$$[R^*(f)](z) = \int\limits_{S^{2n-1}} f(w, \langle z, w\rangle)\, d\sigma(w).$$

It is easy to see that the operator R^* is continuous. It follows from the definition of the Radon transform that

$$\int\limits_{\mathbb{C}^n} [R^*(f)](z)\varphi(z)\, d\omega_{2n}(z) = \int\limits_{\mathbb{C}} \int\limits_{S^{2n-1}} f(w,s)\hat{\varphi}(w,s)\, d\sigma(w) d\omega_2(s) \tag{2}$$

for every function $\varphi \in \mathcal{D}(\mathbb{C}^n)$.

Let $M_{\mathcal{D}}$ be the subspace of $\mathcal{D}(X)$ formed by the functions

$$\psi(w,s) = \frac{\partial^{2n-2}\hat{\varphi}(w,s)}{\partial s^{n-1}\partial \bar{s}^{n-1}}, \quad \hat{\varphi} \in R\mathcal{D}(X). \tag{3}$$

We give $M_{\mathcal{D}}$ the topology induced from $\mathcal{D}(X)$.

Definition 1. Let $F \in \mathcal{D}'$. The Radon transform RF of F is the functional on $M_{\mathcal{D}}$ given by

$$\langle RF, \psi\rangle = \langle F, R^*\psi\rangle. \tag{4}$$

For every function $\varphi \in \mathcal{S}(\mathbb{C}^n)$ the following inversion formula holds [1, p. 118]:

$$\varphi(z) = (-1)^{n-1} c_n R^* \left(\frac{\partial^{2n-2}\hat{\varphi}(w,s)}{\partial s^{n-1}\partial \bar{s}^{n-1}} \right), \tag{5}$$

where $\hat{\varphi}(w,s)$ is the Radon transform of φ, and $c_n > 0$. It follows from the inversion formula (5) that for each function $\psi \in M_{\mathcal{D}}$ the function $R^*(\psi)(z)$ belongs to $\mathcal{D}(\mathbb{C}^n)$. Therefore the functional RF is well defined.

Definition 2. We say that the Radon transform RF of a distribution $F \in \mathcal{D}'$ is defined as a distribution if the functional RF given by (4) can be extended to a continuous functional on $\mathcal{D}(X)$.

It has been shown in [4] that there are distributions in $\mathbb{R}^m$ whose real Radon transforms are not defined as distributions. It is natural to suppose that there are such examples in the case of the complex Radon transform. If the distribution F is given by the function $f(z) \in \mathcal{S}(\mathbb{C}^n)$, then it follows from (5) and (2) that the Radon transform RF is defined as a distribution and it is given by the function $\hat{f}(w,s)$.

We denote by $\mathcal{O}'_C(\mathbb{C}^n)$ the space of rapidly decreasing distributions [5, p. 419]. A distribution $T \in \mathcal{D}'(\mathbb{C}^n)$ belongs to $\mathcal{O}'_C(\mathbb{C}^n)$ if and only if for every $k \in \mathbb{Z}$ the distribution $(1+|x|^2)^k T$ is integrable; i.e.,

$$(1+|x|^2)^k T = \sum_{|p|+|q| \le m(k)} \partial^p \bar{\partial}^q \mu_{pq}(k), \tag{6}$$

where $m(k) \in \mathbb{N}$ and $\{\mu_{pq}\}(k)$ is a finite family of bounded measures on $\mathbb{C}^n$. In particular, every distribution with compact support is rapidly decreasing.

Let $T \in \mathcal{O}'_C(\mathbb{C}^n)$. We show that equality (4) defines the extension of the Radon transform RT to a continuous linear functional on $\mathcal{D}(X)$. Let $h(w,s) \in \mathcal{D}(X)$ be such that $|h(w,s)| \le 1$. There is $R > 0$ such that $h(w,s) = 0$ for $|s| \ge R$, and we have

$$|[R^*h](z)| \leq \int\limits_{S^{2n-1}} |h(w,\langle z,w\rangle)|d\sigma(w) \leq \int\limits_{|\langle z,w\rangle|\leq R} d\sigma(w) \leq d_n \max\left(1, \frac{R^2}{|z|^2}\right), \tag{7}$$

where $d_n > 0$. Suppose that the sequence $\{h_N(w,s)\}$ in $\mathcal{D}(X)$ converges to 0. Then, for every multi-indices p and q, we have

$$\partial^p\bar{\partial}^q \left[R^*(h_N)\right](z) = \int\limits_{S^{2n-1}} \frac{\partial^{|p|+|q|}}{\partial s^{|p|}\partial \bar{s}^{|q|}} h_N(w,\langle z,w\rangle) w^p \bar{w}^q d\sigma(w). \tag{8}$$

There exists $R > 0$ such that $\operatorname{supp}(h_N) \subset S^{2n-1} \times \{s : |s| \leq R\}$ for all N. Then it follows from (7) and (8) that

$$\left|\partial^p\bar{\partial}^q \left[R^*(h_N)\right](z)\right| \leq d_n \max\left(1, \frac{R^2}{|z|^2}\right) \max_{w,s}\left|\frac{\partial^{|p|+|q|}}{\partial s^{|p|}\partial \bar{s}^{|q|}} h_N(w,s)\right|. \tag{9}$$

This means that the functions $\left[R^*(h_N)\right](z)$, together with derivatives of all orders, vanish at infinity. By the definition of the topology of $\mathcal{D}(X)$ we have

$$\lim_{N\to\infty} \max_{w,s}\left|\frac{\partial^{|p|+|q|}}{\partial s^{|p|}\partial \bar{s}^{|q|}} h_N(w,s)\right| = 0. \tag{10}$$

We set $k = 0$ in (6). Then we obtain from (6) and (4) that

$$\langle RT, h_N\rangle = \langle T, [R^*h_N]\rangle = \sum_{|p|+|q|\leq m} (-1)^{|p|+|q|} \int\limits_{\mathbb{C}^n} \partial^p\bar{\partial}^q [R^*h_N](z) d\mu_{pq}(z).$$

Since the measures μ_{pq} are bounded, it follows from (9) and (10) that $\langle RT, h_N\rangle \to 0$ as $N \to \infty$. Thus, for eve ry $T \in \mathcal{O}'_C(\mathbb{C}^n)$, the functional RT is well-defined and continuous on $\mathcal{D}(X)$.

Theorem 1

Let $T \in \mathcal{O}'_C(\mathbb{C}^n)$ and let $K \subset \mathbb{C}^n$ be a linearly convex compact set. Suppose that for every $z \notin K$ there exists a hyperplane $P = \{\lambda : \langle\lambda, w_0\rangle = s_0\}$ satisfying the following conditions:
(i) P contains z.
(ii) P does not meet K.
(iii) The set $\mathbb{C} \setminus K_{w_0}$ is connected, where $K_{w_0} = \{\langle\lambda, w_0\rangle\}_{\lambda\in K}$ is the projection of K on w_0. Then T has support in K if and only if its Radon transform RT has support in $\hat{K}$.

E.T. Quinto [10] has proved the following theorem[1]

Theorem 2

Assume the Radon transform R on complex hyperplanes has a nowhere zero real analytic weight. Let A be an open connected set of complex hyperplanes. Let $f \in$

[1] The author's thanks are due to my referee who referred to Quito's article.

$\mathcal{E}'(\mathbb{C}^n)$ with $Rf(H) = 0$ for all $H \in A$ and assume for some $H_0 \in A$, H_0 is disjoint from supp$f$*. Then, for all $H \in A$, H is disjoint from* supp$f$*.*

If under the hypotheses and notation of Theorem 1 the distribution T belongs to $\mathcal{E}'(\mathbb{C}^n)$, then the proof of Theorem 1 can be reduced to the Theorem 2. The proof of Theorem 1 is based on the reducing to the case of compactly supported distributions (we use the special case of Theorem 2 which was proved by the author [13] independently of Quinto's result.) As usual we use the properties of the convolution $\operatorname{supp} T * \alpha_\varepsilon$ of T and smooth compactly supported functions $\alpha_\varepsilon \in \mathcal{D}(B^n(0,\varepsilon))$. The difficulty is that the compact set

$$K_\varepsilon = \bigcup_{z \in K} \bar{B}^n(z, \varepsilon)$$

may not satisfy the condition (iii) of Theorem 1 which is essential [13]. However it can be shown that supp $(T * \alpha_\varepsilon) \subset \bar{K}_\varepsilon$, where $\bar{K}_\varepsilon$ is the smallest compact set which contains K_ε and satisfies the condition (iii) of Theorem 1. Therefore we have to show that the sets $\bar{K}_\varepsilon$ are correctly defined and $\bar{K}_\varepsilon \to K$ as $\varepsilon \to 0$[2]. However, for our purpose, it is enough to prove a "weak" version of this assertion (Lemma 2 below).

Proof of Theorem 1. Suppose that $T \in \mathcal{O}'_C(\mathbb{C}^n)$ has support in K. Then $T \in \mathcal{E}(\mathbb{C}^n)$. Let $h(w,s) \in \mathcal{D}(X)$ be such that $\operatorname{supp}(h) \subset X \setminus \hat{K}$. If $z \in K$, then the point $(w, \langle z, w\rangle)$ belongs to $\hat{K}$ for every $w \in S^{2n-1}$. Therefore the functions

$$[R^* h](z) = \int_{S^{2n-1}} h(w, \langle z, w\rangle) d\sigma(w),$$

$$\partial^p \bar{\partial}^q \left[R^*(h)\right](z) = \int_{S^{2n-1}} \frac{\partial^{|p|+|q|}}{\partial s^{|p|} \partial \bar{s}^{|q|}} h(w, \langle z, w\rangle) w^p \bar{w}^q d\sigma(w)$$

vanish on K. So $[R^* h](z)$ is an infinitely differentiable function which, together with derivatives of all orders, vanishes on the support of the distribution T. Then we have $\langle T, R^* h\rangle = 0$. Thus, for each $h \in \mathcal{D}(X)$ with $\operatorname{supp}(h) \in X \setminus \hat{K}$ we have $\langle RT, h\rangle = \langle T, [R^* h]\rangle = 0$. This means that $\operatorname{supp}(RT) \subset \hat{K}$.

Before proving the second statement of Theorem 1, we have to show that the dual Radon transform and the convolution operation commute:

Lemma 1

Let $\varphi(z) \in \mathcal{D}(\mathbb{C}^n)$. Then for every $\psi(w,s) \in \mathcal{E}(X)$ the following formula holds:

$$\varphi * [R^* \psi] = R^*[\hat{\varphi} *_s \psi],$$

where $\hat{\varphi}(w,s)$ is the Radon transform of φ, and $*_s$ denotes the convolution with respect to the second variable s.

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[2] The idea to introduce the sets $\bar{K}_\varepsilon$ was proposed by the referee of this article.

Proof. For every function $\alpha(z) \in \mathcal{D}(\mathbb{C}^n)$ we have

$$\int_{\mathbb{C}^n} (\varphi * [R^*\psi])(z)\alpha(z)d\omega_{2n}(z) = \int_{\mathbb{C}^n} [R^*\psi](z)\,(\alpha * \varphi_1)\,(z)d\omega_{2n}(z), \tag{11}$$

where $\varphi_1(z) = \varphi(-z)$. Let J be the integral on the right-hand side of (11). It follows from (2) that

$$J = \int_{S^{2n-1}\times\mathbb{C}} \psi(w,s)\widehat{\alpha * \varphi_1}(w,s)d\sigma(w)d\omega_2(s),$$

where $\widehat{\alpha * \varphi_1}(w,s)$ is the Radon transform of the convolution $\alpha * \varphi$. We have [1, p.p. 116-117]

$$\widehat{\alpha * \varphi_1}(w,s) = (\hat{\alpha} *_s \hat{\varphi}_1)(w,s), \quad \hat{\varphi}_1(w,s) = \hat{\varphi}(-w,s) = \hat{\varphi}(w,-s).$$

Then

$$\begin{aligned} J &= \int_{S^{2n-1}\times\mathbb{C}} \psi(w,s)(\hat{\alpha} *_s \hat{\varphi}_1)(w,s)d\sigma(w)d\omega_2(s) \\ &= \int_{S^{2n-1}\times\mathbb{C}} (\psi *_s \hat{\varphi})(w,s)\hat{\alpha}(w,s)d\sigma(w)d\omega_2(s). \end{aligned}$$

In view of (2), we have

$$J = \int_{\mathbb{C}^n} R^*[\varphi *_s \psi](z)\alpha(z)d\omega_{2n}(z).$$

Then it follows from (11) that

$$\int_{\mathbb{C}^n} \{(\varphi * [R^*\psi])(z) - R^*[\varphi *_s \psi](z)\}\,\alpha(z)d\omega_{2n}(z) = 0$$

for every $\alpha(z) \in \mathcal{D}(\mathbb{C}^n)$. Therefore $(\varphi * [R^*\psi])(z) \equiv R^*[\varphi *_s \psi](z)$. The lemma is proved. □

Now suppose that the support of the Radon transform RT of a distribution $T \in \mathcal{O}'_C(\mathbb{C}^n)$ is contained in $\hat{K}$. Let $\{\alpha_m(z)\}_{m=1}^{\infty}$ be a sequence of smooth functions on $\mathbb{C}^n$ with $\mathrm{supp}(\alpha_m) \subset \{z : |z| \le 1/m\}$ that converges in the space of measures to the delta function at the origin. We assume that the functions $\alpha_m(z)$ are even, i.e., $\alpha_m(-z) = \alpha_m(z)$. We set $T_m = T * \alpha_m$. Then the function $T_m(z)$ belongs to $\mathcal{S}(\mathbb{C}^n)$ [12, p. 244], and $T_m \to T$ in $\mathcal{O}'_C(\mathbb{C}^n)$ [4]. Denote by K_m the compact set

$$K_m = \bigcup_{z\in K} \bar{B}^n(z, 1/m).$$

Let $\hat{T}_m(w,s)$ be the Radon transform of $T_m(z)$. We show that $\mathrm{supp}(\hat{T}_m) \subset \hat{K}_m$. The hyperplane $\{z : \langle z, w\rangle = s\}$ meets K_m if and only if there are $z' \in K$, $z'' \in \bar{B}^n(0, 1/m)$ such that $\langle z', w\rangle = s - \langle z'', w\rangle$. Therefore

$$\hat{K}_m = \bigcup_{(w,s)\in\hat{K}} \left(\{w\} \times \bar{B}^1(s, 1/m)\right). \tag{12}$$

Let $h(w,s) \in \mathcal{D}(S^{2n-1} \times \mathbb{C})$ be such that $\operatorname{supp}(h) \cap \hat{K}_m = \emptyset$. Since the functions α_m are even, it follows from (4) that

$$\langle RT_m, h\rangle = \langle T_m, R^*(h)\rangle = \langle T * \alpha_m, R^*(h)\rangle = \langle T, \alpha_m * R^*(h)\rangle.$$

Then by Lemma 1, we have $\langle T, \alpha_m * R^*(h)\rangle = \langle T, R^*(\hat{\alpha}_m *_s h)\rangle$. Then

$$\langle RT_m, h\rangle = \langle T, R^*(\hat{\alpha}_m *_s h)\rangle = \langle RT, \hat{\alpha}_m *_s h\rangle. \tag{13}$$

We claim that $\hat{K} \cap \operatorname{supp}(\hat{\alpha}_m *_s h) = \emptyset$. Indeed, suppose that $(w_0, s_0) \in \hat{K} \cap \operatorname{supp}(\hat{\alpha}_m *_s h)$. This implies (since $\hat{\alpha}_m(w,s) = 0$ for $|s| \geq 1/m$) that for some $s_1 \in \bar{B}^1(0, 1/m)$ we have $(w_0, s_0 + s_1) \in \operatorname{supp}(h)$. By (12) we also have $(w_0, s_0 + s_1) \in \hat{K}_m$, which contradicts that $\operatorname{supp}(h) \cap \hat{K}_m = \emptyset$. Therefore $\hat{K} \cap \operatorname{supp}(\hat{\alpha}_m *_s h) = \emptyset$, and it follows from (13) (since $\operatorname{supp}(RT) \subset \hat{K}$) that $\langle RT_m, h\rangle = 0$. Therefore

$$\operatorname{supp}(RT_m) \subset \hat{K}_m. \tag{14}$$

As remarked above, the functions $T_m(z)$ belong to $\mathcal{S}(\mathbb{C}^n)$. Then the distributions RT_m are given by the Radon transforms $\hat{T}_m(w,s)$ of functions $T_m(z)$.

In view of (12), there exist $R > 0$ such that for all m the sets $\hat{K}_m$ are contained in the set $\{(w,s) : |s| \leq R\}$. Let $R_{\mathbb{R}}T_m(w,t)$ be the real Radon transform of $T_m(z)$, that is

$$R_{\mathbb{R}}T_m(w,t) = \int\limits_{\operatorname{Re}\langle z, \bar{w}\rangle = t} T_m(z)\, d\lambda(z),$$

where $d\lambda(z)$ is the area element on the real hyperplane $\{z : \operatorname{Re}\langle z, \bar{w}\rangle = t\}$. Then we have

$$R_{\mathbb{R}}T_m(w,t) = \int_{-\infty}^{\infty} \hat{T}_m(\bar{w}, t + ix)\, dx.$$

Since $\hat{K}_m \subset \{(w,s) : |s| \leq R\}$, it follows from (14) that $R_{\mathbb{R}}T_m(w,t) = 0$ for $|t| \geq R$. Then by the Helgason's support theorem, the supports of the functions $T_m(z)$ are compact.

To complete the proof of Theorem 1, we need the following lemma:

Lemma 2

Under the hypotheses and notation of Theorem 1, there exist, for every $z_0 \notin K$, a neighborhood V_{z_0} and $\delta > 0$ such that the functions $T_m(z)$ vanish on V_{z_0} for $m \geq 1/\delta$.

Proof. Fix $z_0 \notin K$. Then there exists a point $(w_0, s_0) \in S^{2n-1} \times \mathbb{C}$ such that $\{z : \langle z, w_0\rangle = s_0\} \cap K = \emptyset$, $\langle z_0, w_0\rangle = s_0$ and the set $\mathbb{C} \setminus \{\langle z, w_0\rangle\}_{z \in K}$ is connected. Then $(w_0, \langle z_0, w_0\rangle) \notin \hat{K}$. We set

$$A = \left\{ s \in \mathbb{C} \mid (w_0, s) \in \hat{K} \right\}, \quad A_m = \left\{ s \in \mathbb{C} \mid (w_0, s) \in \hat{K}_m \right\}.$$

It follows from (12) that

$$A_m = \bigcup_{s \in A} \bar{B}^1(s, 1/m).$$

By definition of $\hat{K}$, for every $s \in A$ there exists $z \in K$ such that $\langle z, w_0 \rangle = s$. Then $A = \{\langle z, w_0 \rangle\}_{z \in K}$. Similarly $A_m = \{\langle z, w_0 \rangle\}_{z \in K_m}$. Since the sets K and K_m are compact, it follows that the sets A and A_m are also compact. For some $R > 0$ we have $A \cup A_m \subset \bar{B}^1(0, R)$. Since $\langle z_0, w_0 \rangle \notin A$, there is $\gamma > 0$ such that $\langle z_0 + \lambda, w_0 \rangle \notin A$ for every $\lambda \in \bar{B}^n(0, \gamma)$. Hence the convex compact set $\Gamma_1 = \left\{\langle z, w_0 \rangle, z \in \bar{B}^n(z_0, \gamma)\right\}$ and the set A do not intersect. Fix $s_1 \in \{s \in \mathbb{C} : |s| > R\}$. Then $s_1 \in \mathbb{C} \setminus A$. Since the set $\mathbb{C} \setminus A$ is connected, there exists a broken line $\Gamma_2 \subset \mathbb{C} \setminus A$ joining s_1 to the point $\langle z_0, w_0 \rangle$. Thus $(\Gamma_1 \cup \Gamma_2) \cap A = \emptyset$. Then, since the sets $\Gamma_1 \cup \Gamma_2$ and A are compact, there exists $\delta \in (0, 1)$ such that for all $m \geq 1/\delta$ we have

$$\left\{(\Gamma_1 \cup \Gamma_2) + B^1(0, \delta)\right\} \cap \left\{A + \bar{B}^1\left(0, 1/m\right)\right\} = \emptyset,$$

that is $\left\{(\Gamma_1 \cup \Gamma_2) + B^1(0, \delta)\right\} \cap A_m = \emptyset$. Put

$$D = \{s \in \mathbb{C} : |s| > R\} \cup \left\{(\Gamma_1 \cup \Gamma_2) + B^1(0, \delta)\right\}.$$

By construction D is a connected unbounded open set containing the point $\langle z_0 + \lambda, w_0 \rangle$ for every $\lambda \in \bar{B}^n(0, \gamma)$. We have by the definition of the sets A_m that $(D \times \{w_0\}) \cap \hat{K}_m = \emptyset$ for $m \geq 1/\delta$. Then it follows from (14) that $(D \times \{w_0\}) \cap \operatorname{supp}(\hat{T}_m) = \emptyset$ for $m \geq 1/\delta$. Since the supports of T_m are compact, it follows from [13, Theorem 2] that for every $\lambda \in \bar{B}^n(0, \gamma)$ and $m \geq 1/\delta$ the functions $T_m(z)$ vanish on the hyperplane $\{z : \langle z, w_0 \rangle = \langle z_0 + \lambda, w_0 \rangle\}$. Then, for every $z \in \bar{B}^n(z_0, \gamma)$ and $m \geq 1/\delta$, we have $T_m(z) = 0$. The lemma is proved. $\square$

As mentioned above, $T_m \to T$ in $\mathcal{O}'_C(\mathbb{C}^n)$. This means that

$$\lim_{m \to \infty} \langle T_m, \varphi \rangle = \langle T, \varphi \rangle, \quad \forall \varphi \in \mathcal{O}_C(\mathbb{C}^n), \tag{15}$$

where $\mathcal{O}_C(\mathbb{C}^n)$ is the space of all infinitely differentiable functions f on $\mathbb{C}^n$ for which there exist an integer k such that $(1+|x|^2)^k \partial^p \bar{\partial}^q f(z)$ vanishes at infinity for all p, q [5, p. 173]. Since $\mathcal{D}(\mathbb{C}^n) \subset \mathcal{O}_C(\mathbb{C}^n)$, formula (15) holds for every $\varphi \in \mathcal{D}(\mathbb{C}^n)$. Let $\varphi \in \mathcal{D}(\mathbb{C}^n)$ be such that $\operatorname{supp}(\varphi) \cap K = \emptyset$. By Lemma 2 for every $z \in \operatorname{supp}\varphi$ there are $\delta(z) > 0$ and a ball $B^n(z, \gamma(z))$ such that $T_m(z) = 0$ on $B^n(z, \gamma(z))$ for $m \geq 1/\delta(z)$. Since the support of φ is compact, it can be covered by a finite union of balls $B^n(z_k, \gamma(z_k))$, where $k = 1, 2 \ldots, N$. Setting $\delta_0 = \min\{\delta(z_k), 1 \leq k \leq N\}$, we have $T_m(z) = 0$ for $z \in \operatorname{supp}(\varphi)$ and $m \geq 1/\delta_0$. Then it follows from (15) that

$$\langle T, \varphi \rangle = \lim_{m \to \infty} \langle T_m, \varphi \rangle = 0.$$

Since $\varphi \in \mathcal{D}(\mathbb{C}^n)$ is an arbitrary function such that $\operatorname{supp}(\varphi) \cap K = \emptyset$, we have $\operatorname{supp}(T) \subset K$. The theorem is proved. $\square$

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