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Asymptotically isometric copies of c_0 and ℓ^1 in quotients of Banach spaces

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Abstract

Let X be a real Banach space that does not contain a copy of ℓ^1. Then X^* contains asymptotically isometric copies of ℓ^1 if and only if X has a quotient which is asymptotically isometric to c_0.

1. Introduction

It is a classical result of W.B. Johnson and H.P. Rosenthal [8] that if X is a separable Banach space, then X^* contains a subspace isomorphic to ℓ^1 if and only if X has a quotient isomorphic to c_0. In [1], an asymptotically isometric version of this result was proved, that is, if X is a separable Banach space, then X^* contains asymptotically isometric copies of ℓ^1 if and only if X has a quotient which is asymptotically isometric to c_0. The notion of an asymptotically isometric copy of ℓ^1(resp. c_0) was introduced in [5] (resp. [6]) and used to show that some spaces fail the fixed point property for non-expansive self-maps on closed bounded convex sets. Another well-known result of J. Hagler and W.B. Johnson [7] says that if a real Banach space X does not contain a copy of ℓ^1, then X^* contains a subspace isomorphic to ℓ^1 if and only if X has a quotient isomorphic to c_0. The main result of this note states that as in the isomorphic case, if a real Banach space X does not contain a copy of ℓ^1, then X^* contains asymptotically isometric copies of ℓ^1 if and only if X has a quotient which is asymptotically isometric to c_0.

Our notation and terminology are standard as may be found in [9], [3].

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2. Definitions and results

DEFINITION 2.1 ([5]). A Banach space X is said to contain an asymptotically isometric copy of ℓ^1 if there is a null sequence $(\epsilon_n)_n$ in (0,1) and a sequence $(x_n)_n$ in X such that

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \leq \Big\| \sum_{n=1}^{\infty} t_n x_n \Big\| \leq \sum_{n=1}^{\infty} |t_n|$$

for all $(t_n)_n \in \ell^1$. We will refer to the sequence $(x_n)_n$ as an asymptotically isometric ℓ^1-sequence.

DEFINITION 2.2 ([6]). A Banach space X is said to contain an asymptotically isometric copy of c_0 if there is a null sequence $(\epsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X such that

$$\sup_n (1-\epsilon_n)|t_n| \leq \Big\| \sum_{n=1}^{\infty} t_n x_n \Big\| \leq \sup_n |t_n|$$

for all $(t_n)_n \in c_0$.

We say that a Banach space X is asymptotically isometric to c_0 if X has a basis $(x_n)_n$ with the above property.

Before giving the main result, we prove the following result which is of independent interest.

Theorem 2.1

Let X be a Banach space. Then X^* contains a weak*-null asymptotically isometric ℓ^1-sequence if and only if X has a quotient which is asymptotically isometric to c_0.

Proof. (Necessity) Suppose that $(x_n^*)_n$ is a *weak**-null asymptotically isometric ℓ^1-sequence in X^*. Then there is a null sequence $(\epsilon_n)_n$ in (0,1) such that

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \leq \Big\| \sum_{n=1}^{\infty} t_n x_n^* \Big\| \leq \sum_{n=1}^{\infty} |t_n|$$

for all $(t_n)_n \in \ell^1$. Define $T : X \longrightarrow c_0$ by $T(x) = (x_n^*(x))_n$ for all $x \in X$. Then $T^*(t) = \sum_{n=1}^{\infty} t_n x_n^*$ for all $t = (t_n)_n \in \ell^1$. Thus T^* is an isomorphism into. This implies that T is onto. Define $\widehat{T} : X/\mathrm{Ker}(T) \longrightarrow c_0$ by $\widehat{T}([x]) = T(x)$ for all $[x] \in X/\mathrm{Ker}(T)$. Therefore $\widehat{T}$ is an isomorphism onto. For each $n \in \mathbb{N}$, choose $z_n \in X/\mathrm{Ker}(T)$ with $\widehat{T}(z_n) = e_n$, where $(e_n)_n$ is the standard unit vector basis in c_0. Hence $(z_n)_n$ is a shrinking basis for $X/\mathrm{Ker}(T)$. Define $z_n^*(\sum_{k=1}^{\infty} a_k z_k) = a_n$, for all $\sum_{k=1}^{\infty} a_k z_k \in X/\mathrm{Ker}(T)$. Then $(z_n^*)_n$ is a basis for $(X/\mathrm{Ker}(T))^*$. It can be checked that $\mathrm{Ker}(T) = (\overline{\mathrm{span}}\{x_n^* : n \in \mathbb{N}\})^{\top}$ and $Q^*(z_n^*) = x_n^*$ for all $n \in \mathbb{N}$ (in fact we have shown that $(x_n^*)_n$ is a w^*-basic sequence), where $Q : X \longrightarrow X/\mathrm{Ker}(T)$ is the quotient mapping. Next we show that $Y = X/\mathrm{Ker}(T)$ is asymptotically isometric to c_0. For all $(t_n)_n \in c_0$, consider $z = \sum_{n=1}^{\infty} t_n z_n$. Then, for each $n \in \mathbb{N}$, $t_n = z_n^*(z)$. By the Hahn-Banach Theorem, there is a $z^* \in Y^*$ such that $\|z^*\| = 1$ and $z^*(z) = \|z\|$. Since $Q^* : Y^* \longrightarrow X^*$ is a linear isometry into, we have

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \le \Big\|\sum_{n=1}^{\infty} t_n z_n^*\Big\| \le \sum_{n=1}^{\infty}|t_n|,$$

for all $(t_n)_n \in \ell^1$. Since $(z_n^*)_n$ is a basis for Y^*, $z^* = \sum\limits_{n=1}^{\infty} z^*(z_n)z_n^*$. Thus

$$\begin{aligned}
\Big\|\sum_{n=1}^{\infty} z_n^*(z)z_n\Big\| &= \sum_{n=1}^{\infty} z_n^*(z)z^*(z_n)\\
&\le \sum_{n=1}^{\infty} |z_n^*(z)||z^*(z_n)|\\
&= \sum_{n=1}^{\infty} \frac{1}{1-\epsilon_n}|z_n^*(z)|(1-\epsilon_n)|z^*(z_n)|\\
&\le \Big(\sup_n \frac{1}{1-\epsilon_n}|z_n^*(z)|\Big)\sum_{n=1}^{\infty}(1-\epsilon_n)|z^*(z_n)|\\
&\le \Big(\sup_n \frac{1}{1-\epsilon_n}|z_n^*(z)|\Big)\Big\|\sum_{n=1}^{\infty} z^*(z_n)z_n^*\Big\|\\
&= \sup_n \frac{1}{1-\epsilon_n}|z_n^*(z)|.
\end{aligned}$$

On the other hand, for each $k \in \mathbb{N}$,

$$\Big|z_k^*\Big(\sum_{n=1}^{\infty} z_n^*(z)z_n\Big)\Big| \le \Big\|\sum_{n=1}^{\infty} z_n^*(z)z_n\Big\|,$$

and hence

$$\sup_n |z_n^*(z)| \le \Big\|\sum_{n=1}^{\infty} z_n^*(z)z_n\Big\|.$$

Thus,

$$\sup_n |z_n^*(z)| \le \Big\|\sum_{n=1}^{\infty} z_n^*(z)z_n\Big\| \le \sup_n \frac{1}{1-\epsilon_n}|z_n^*(z)|.$$

That is,

$$\sup_n |t_n| \le \Big\|\sum_{n=1}^{\infty} t_n z_n\Big\| \le \sup_n \frac{1}{1-\epsilon_n}|t_n|.$$

Let $y_n = (1-\epsilon_n)z_n$, for all $n \in \mathbb{N}$. Then $(y_n)_n$ is a basis for Y, and moreover, for all $(t_n)_n \in c_0$, we have

$$\sup_n(1-\epsilon_n)|t_n| \le \Big\|\sum_{n=1}^{\infty} t_n y_n\Big\| \le \sup_n |t_n|.$$

This completes the proof of the necessity of Theorem 2.1.

(Sufficiency) Assume that $X/M(M$ is a closed subspace of $X)$ is asymptotically isometric to c_0. Then there is a null sequence $(\epsilon_n)_n$ in $(0,1)$ and a basis $([x_n])_n$ in X/M such that

$$\sup_n (1-\epsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n[x_n] \right\| \leq \sup_n |t_n|$$

for all $(t_n)_n \in c_0$. Define $[x_n]^*$ on $(X/M)^*$ by $[x_n]^*(\sum_{k=1}^{\infty} t_k[x_k]) = t_n$, for all $\sum_{k=1}^{\infty} t_k[x_k] \in X/M$. For each $n \in \mathbb{N}$,

$$\left| [x_n]^* \Big(\sum_{k=1}^{\infty} t_k[x_k] \Big) \right| = |t_n| = \frac{1}{1-\epsilon_n}(1-\epsilon_n)|t_n| \leq \frac{1}{1-\epsilon_n} \left\| \sum_{k=1}^{\infty} t_k[x_k] \right\|.$$

Thus $\|[x_n]^*\| \leq \frac{1}{1-\epsilon_n}$, for all $n \in \mathbb{N}$. Let $\omega_n^* = \frac{[x_n]^*}{\|[x_n]^*\|}$. Then for all scalars $t_1, t_2, ..., t_m$ and for all $m \in \mathbb{N}$, we have $\| \sum_{n=1}^{m} t_n \omega_n^* \| \leq \sum_{n=1}^{m} |t_n|$. On the other hand, since $\| \sum_{n=1}^{m} \operatorname{sgn}(t_n)[x_n] \| \leq 1$,

$$\begin{aligned} \left\| \sum_{n=1}^{m} t_n \omega_n^* \right\| &\geq \left| \Big(\sum_{n=1}^{m} t_n \omega_n^* \Big) \Big(\sum_{n=1}^{m} \operatorname{sgn}(t_n)[x_n] \Big) \right| \\ &= \sum_{n=1}^{m} |t_n| \frac{1}{\|[x_n]^*\|} \\ &\geq \sum_{n=1}^{m} (1-\epsilon_n)|t_n|. \end{aligned}$$

That is,

$$\sum_{n=1}^{m} (1-\epsilon_n)|t_n| \leq \left\| \sum_{n=1}^{m} t_n \omega_n^* \right\| \leq \sum_{n=1}^{m} |t_n|.$$

Hence $(\omega_n^*)_n$ is an asymptotically isometric ℓ_1-sequence in $(X/M)^*$. Since $Q^* : (X/M)^* \longrightarrow X^*$ is a linear isometry into, $(Q^*(\omega_n^*))_n$ is also an asymptotically isometric ℓ_1-sequence in X^*. It is easy to check that $(Q^*(\omega_n^*))_n$ is a *weak**-null sequence in X^*. This completes the proof. □

Remark 2.1. The main result in [1], Theorem 1, can be easily obtained by the above result. Indeed, if a Banach space X is separable and X^* contains asymptotically isometric copies of ℓ^1, it is easy to construct a *weak**-null asymptotically isometric ℓ^1-sequence in X^*.

To complete the proof of our main result, Theorem 2.4, we need the following two results.

Theorem 2.2 ([7], [2])

Let X be a real Banach space, and let $(x_n^*)_n$ be a sequence in X^* equivalent to the unit vector basis of ℓ^1. If no normalized ℓ^1-block of $(x_n^*)_n$ is weak*-null sequence, then X contains a copy of ℓ^1.

Theorem 2.3 ([4])

If a Banach space X contains an asymptotically isometric copy of c_0, then X^* contains an asymptotically isometric copy of ℓ^1.

Theorem 2.4

Let X be a real Banach space that does not contain a copy of ℓ^1. Then X^* contains asymptotically isometric copies of ℓ^1 if and only if X has a quotient which is asymptotically isometric to c_0.

Proof. (Necessity) Since X^* contains asymptotically isometric copies of ℓ^1, there is a null sequence $(\epsilon_n)_n$ in (0,1) and a sequence $(x_n^*)_n$ in X^* such that

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \leq \Big\|\sum_{n=1}^{\infty} t_n x_n^*\Big\| \leq \sum_{n=1}^{\infty}|t_n|$$

for all $(t_n)_n \in \ell^1$. According to the Theorem 2.2, there is a $weak^*$-null sequence $(z_n^*)_n$ which is a normalized ℓ^1-block of $(x_n^*)_n$, that is, $z_n^* = \sum_{k\in A_n} a_k x_k^*$, where $(A_n)_n$ is a sequence of pairwise disjoint finite subsets of $\mathbb{N}$, $A_n < A_{n+1}$, and $\sum_{k\in A_n}|a_k| = 1$, for all $n \in \mathbb{N}$. Then for all scalars $t_1, t_2, ..., t_m$ and all $m \in \mathbb{N}$, we have

$$\begin{aligned}
\Big\|\sum_{n=1}^{m} t_n z_n^*\Big\| &= \Big\|\sum_{n=1}^{m} t_n\Big(\sum_{k\in A_n} a_k x_k^*\Big)\Big\| \\
&\leq \sum_{n=1}^{m}|t_n|\Big(\sum_{k\in A_n}|a_k|\Big) \\
&= \sum_{n=1}^{m}|t_n|.
\end{aligned}$$

On the other hand, for each $n \in \mathbb{N}$, choose $k_n \in A_n$ with $1-\epsilon_{k_n} = \min_{k\in A_n}(1-\epsilon_k)$. Then

$$\begin{aligned}
\Big\|\sum_{n=1}^{m} t_n z_n^*\Big\| &= \Big\|\sum_{n=1}^{m} t_n\Big(\sum_{k\in A_n} a_k x_k^*\Big)\Big\| \\
&\geq \sum_{n=1}^{m}|t_n|\Big(\sum_{k\in A_n}(1-\epsilon_k)|a_k|\Big) \\
&\geq \sum_{n=1}^{m}(1-\epsilon_{k_n})|t_n|.
\end{aligned}$$

Thus $(z_n^*)_n$ is a $weak^*$-null asymptotically isometric ℓ^1-sequence in X^*. It follows from Theorem 2.1 that X has a quotient which is asymptotically isometric to c_0.

(Sufficiency) Suppose that $X/M(M$ is a closed subspace of $X)$ is asymptotically isometric to c_0. Then, by Theorem 2.3, $(X/M)^* = M^{\perp}$ contains an asymptotically isometric copy of ℓ^1. Thus X^* contains an asymptotically isometric copy of ℓ^1. This completes the proof. □

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References

1. Dongyang Chen, Asymptotically isometric copies of c_0 and ℓ^1 in Certain Banach spaces, *J. Math. Anal. Appl.* **284** (2003), 618–625.
2. S. Díaz and A. Fernández, Reflexivity in Banach lattices, *Arch. Math.* **63** (1994), 549–552.
3. J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics, **92**, Springer-Verlag, New York-Berlin, 1984.
4. P.N. Dowling, W.B. Johnson, C.J. Lennard, and B. Turett, The optimality of James's distortion theorems, *Proc. Amer. Math. Soc.* **125** (1997), 167–174.
5. P.N. Dowling and C.J. Lennard, Every nonreflexive subspace of $L_1[0,1]$ fails the fixed point property, *Proc. Amer. Math. Soc.* **125** (1997), 443–446.
6. P.N. Dowling, C.J. Lennard, and B. Turett, Reflexivity and the fixed-point property for non-expansive maps, *J. Math. Anal. Appl.* **200** (1996), 653–662.
7. J. Hagler and W.B. Johnson, On Banach spaces whose dual balls are not *weak** sequentially compact, *Israel J. Math.* **28** (1977), 325–330.
8. W.B. Johnson and H.P. Rosenthal, On *w** basic sequences and their applications to the study of Banach spaces, *Studia Math.* **43** (1972), 77–92.
9. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Springer, Berlin, 1977.