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On the osculatory behaviour of higher dimensional projective varieties

Edoardo Ballico and Claudio Fontanari

Università degli Studi di Trento, Dipartimento di Matematica
Via Sommarive 14, 38050 Povo (Trento), Italy

E-mail: ballico@science.unitn.it fontanar@science.unitn.it

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Abstract

We explore the geometry of the osculating spaces to projective varieties of arbitrary dimension. In particular, we classify varieties having very degenerate higher order osculating spaces and we determine mild conditions for the existence of inflectionary points.

1. Introduction

The geometry of osculating spaces to projective varieties is a very classical but still widely open subject. As is well-known, while the dimension of the tangent space at a smooth point is always equal to the dimension of the variety, higher order osculating spaces can be strictly smaller than expected also at a general point.

The investigation of algebraic surfaces having defective second order osculating space was inaugurated in 1907 by Corrado Segre in his seminal paper [17]. The early developments in the field are witnessed by a long series of nice contributions, among which we cannot resist to mention at least [19] by Alessandro Terracini, [5] by Enrico Bompiani, [20] and [21] by Eugenio Togliatti. We point out that the work of Togliatti has been recently reconsidered by several authors: the main result of [21] is reproved in [9] and [7], while the crucial example in [20] is analized in [14] and [12]. However, the modern approach to the subject is mainly concerned with the osculatory behaviour at every (not just at the general) point: see for instance [18], [8], [16], [3], [11].

Here we address the case of projective varieties of arbitrary dimension from various points of view. Namely, in Section 2 we apply the classical approach of Bompiani and we obtain a rough classification of projective varieties having very degenerate higher order osculating spaces (see Theorem 2). In Sections 3 and 4, instead, we apply the

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modern theory of vector bundles and we find explicit conditions assuring the existence of inflectionary points on projective varieties (see Theorems 4 and 5). Other related results are collected in Propositions 1, 2, and 3.

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Let $X \subset \mathbb{P}^r$ be an integral nondegenerate projective variety of dimension n defined over the field $\mathbb{C}$. Let $p \in X$ be a smooth point and consider a lifting

$$\begin{array}{rcl} U \subseteq \mathbb{C}^n & \longrightarrow & \mathbb{C}^{r+1} \setminus \{0\} \\ t & \longmapsto & p(t) \end{array}$$

of a local regular parametrization of X centered in p. The osculating space $T(m,p,X)$ of order m at $p \in X$ is by definition the linear span of the points $[p_I(0)] \in \mathbb{P}^r$, where I is a multi-index such that $|I| \leq m$. We say that X satisfies a differential equation of order m at p if

$$\sum_{|I| \leq m} a_I(t_1, \ldots, t_n) p_I(t_1, \ldots, t_n) = 0$$

in U, with $a_I(t_1, \ldots, t_n) \neq 0$ for some I with $|I| = m$. Hence we have

$$\dim T(m,p,X) = \binom{n+m}{n} - N - 1$$

where N is the number of independent differential equations of order m satisfied by X at p. In [2] we proved the following generalization of a classical result by Bompiani:

Theorem 1 ([2])

Let $X \subset \mathbb{P}^r$ be a smooth variety and let $p \in X$ be a general point. Assume that $\dim T(m,p,X) = h$ and $\dim T(m+1,p,X) = h+k$ with $1 \leq k \leq n-1$. Then either $X \subset \mathbb{P}^{h+k}$ or X is covered by infinitely many subvarieties Y of dimension at least $n-k$ such that $Y \subset \mathbb{P}^{h-m}$.

As an application in the spirit of Bompiani's paper [5], we are going to classify smooth projective varieties satisfying many differential equations at a general point.

Remark 1. Fix $p \in X$ and let $V_m(p)$ be the $\mathbb{C}$-vector space of the differential equations of order $\leq m$ satisfied by X at p. Set $W_m(p) = V_m(p)/V_{m-1}(p)$ and define $h_m(p) := \dim W_m(p)$. There is a natural linear map

$$\begin{array}{rcl} H^0(\mathbb{P}^{n-1}, \mathcal{O}(d)) \otimes W_m(p) & \longrightarrow & W_{m+d}(p) \\ \left(\dfrac{\partial}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}, \{f = 0\} \right) & \longmapsto & \left\{ \dfrac{\partial f}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}} = 0 \right\} \end{array}$$

where $\alpha_1 + \ldots + \alpha_n = d$. By the Hopf Theorem (see for instance [1], p. 108), we have

$$h_{m+d}(p) \geq h_m(p) + \binom{n-1+d}{n-1} - 1.$$

Theorem 2

Let X be a smooth projective variety of dimension n. Assume that X satisfies

$$N \geq (m-1)\left[\binom{n+m-1}{n-1} - n - 1\right] - \sum_{d=1}^{m-2} \binom{n-1+d}{n-1}$$

independent differential equations of order m at a general point $p \in X$. Then either $X \subset \mathbb{P}^{\binom{n+m}{n}-N-1}$ or X is covered by infinitely many subvarieties Y of dimension at least $n-k \geq 1$ such that $Y \subset \mathbb{P}^{\binom{n+m}{n}-N-k-m}$.

Proof. We split the proof into three cases, according to the possible values of $h_m(p)$.

If

$$h_m(p) = \binom{n+m}{n} - \binom{n+m-1}{n} = \binom{n+m-1}{n-1},$$

then $T(m,p,X) = T(m-1,p,X)$. From [6], Proposition 2.3, it follows that $X \subset \mathbb{P}^{\binom{n+m}{n}-N-1}$.

If $h_m(p) = \binom{n+m-1}{n-1} - k$ with $1 \leq k \leq n-1$, then $\dim T(m,p,X) = \dim T(m-1,p,X)+k$. From Theorem 1 it follows that either $X \subset \mathbb{P}^{\binom{n+m}{n}-N-1}$ or X is covered by infinitely many subvarieties Y of dimension at least $n-k$ such that $Y \subset \mathbb{P}^{\binom{n+m}{n}-N-k-m}$.

Assume finally $h_m(p) \leq \binom{n+m-1}{n-1} - n$. From Remark 1 it follows that

$$h_{m-d} \leq \binom{n+m-1}{n-1} - n - \binom{n-1+d}{n-1} + 1$$

for every $d \leq m-2$. Therefore X satisfies at most

$$(m-1)\binom{n+m-1}{n-1} - (m-1)n + (m-2) - \sum_{d=1}^{m-2} \binom{n-1+d}{n-1}$$

equations and this contradiction ends the proof. □

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Let X be an integral n-dimensional projective variety defined over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. Fix a proper closed subset T of X, $L \in \mathrm{Pic}(X)$, and $V \subseteq H^0(X,L)$ such that V spans L. Hence V induces a morphism $\phi_V : X \to \mathbb{P}(V^*) \cong \mathbb{P}^r$, with $r := \dim(V) - 1$. Set $U := X \backslash T$. We always assume that U is smooth and that $\phi_V|U$ is an embedding. We aim to find conditions on X, L, V and T which force the existence of $q \in \phi_V(U)$ which is a hyperosculating point of the variety $\phi_V(X)$. Of course if $T = \emptyset$, then X is smooth and ϕ_V is an embedding. For any integer $m \geq 0$, let $P^m(L)$ be the sheaf of principal parts of order ar most m of L ([15], §2 and §6). Since U is smooth and n-dimensional, $P^m(L)|U$ is locally free of rank $\binom{n+m}{n}$. There is a Taylor series map $a^m : V \otimes \mathcal{O}_X \to P^m(L)$ and the sheaves $\mathrm{Im}(a^m)$ and $\mathrm{Coker}(a^m)$ measure the osculating behaviour up to order m of the variety $\phi_V(X) \subset \mathbb{P}^r$. In particular, if $T(m,p,X)$ is the m-osculating space as defined in the previous section, we have $T(m,p,X) = \mathbb{P}(\mathrm{Im}(a^m(p))) \subseteq \mathbb{P}(V^*)$. We say that $q \in U$ is a hyperosculating point of order at most m for V if the sheaf $\mathrm{Coker}(a^m)$ is not locally free at q; we use the convention that $q \in U$ is not a hyperosculating point of order at most m for V if $\mathrm{Coker}(a^m)$ is the zero-sheaf in a neighborhood of q. Since U is reduced, the set of hyperosculating points is a proper closed subset of U.

Remark 2. Let X be a reduced algebraic variety and $\mathcal{F}$ a coherent sheaf on X. For every $p \in X$ set $\alpha(p) = \dim_{\mathbb{K}} \mathcal{F}_p/\mathfrak{m}_p\mathcal{F}_p$, where $\mathcal{F}_p$ is the stalk of $\mathcal{F}$ at p and $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{X,p}$. It is easy to check that the function $\alpha : X \to \mathbb{N}$ is locally constant if and only if $\mathcal{F}$ is locally free. It follows that if $X \subset \mathbb{P}^n$ and $q \in X_{\mathrm{reg}}$, then q is hyperosculating if and only if there is $m > 0$ such that $\dim T(m, q, X) < \dim T(m, p, X)$ for a general $p \in X$.

Set $x(m) := \mathrm{rank}(\mathrm{Im}(a_m))$. The main result of [8] can be formulated as follows:

Theorem 3 ([8])

Assume $T = \emptyset$, $V = H^0(X, L)$ and $r + 1 = \binom{n+m}{m}$. If $x(m) = r + 1$ and there is no hyperosculating point of order at most m, then $X \cong \mathbb{P}^n$ and $L \cong \mathcal{O}_{\mathbb{P}^n}(m)$.

Our set-up suggests the following natural generalization:

Proposition 1

Assume X smooth, $r + 1 = \binom{n+m}{m} = x(m)$, $\mathrm{codim}(T) \geq 2$ and that there is no hyperosculating point of order at most m on U, then $X \cong \mathbb{P}^n$ and $L \cong \mathcal{O}_{\mathbb{P}^n}(m)$.

Proof. By assumption there is an everywhere injective map (with locally free or zero cokernel) $a^m|U : V \otimes \mathcal{O}_U \to P^m(L)|U$. Since X is reduced, this implies the injectivity of a^m as a map of sheaves. Since X is smooth, $P^m(L)$ is locally free. An injective map of sheaves between two locally free sheaves with the same rank is either an isomorphism or its cokernel is supported exactly by an effective Cartier divisor (use the determinant). Hence a^m is an isomorphism. By Theorem 3, $X \cong \mathbb{P}^n$ and ϕ_V is a Veronese embedding. □

Next we introduce a deeper result which points to the same direction. For any vector bundle E on X (or on U), let $c_*(E)$ denote its total Chern class in the Chow ring of X (resp. U). Since U is smooth, for every integer $t > 0$ we have the following exact sequence on U:

$$0 \to S^t(\Omega^1_U) \to P^t(L)|U \to P^{t-1}(L)|U \to 0 \tag{1}$$

Hence by induction on m one can compute the Chern classes on U (not on X) of $P^m(L)|U$ in terms of $c_1(L|U)$ and the Chern classes of Ω^1_U, i.e. the Chern classes of U.

Theorem 4

Assume $x(m) = r + 1 \leq \binom{n+m}{n} \leq r + n$, and that there is no hyperosculating point of order at most m on U. Then $c_t(P^m(L)|U) = 0$ for every integer t such that $\binom{n+m}{n} - r - 1 < t \leq n$.

Proof. Set $A_U := \mathrm{Coker}(a^m|U)$. Since $x(m) = r+1$, a^m is generically injective. Since X is reduced and $P^m(L)|U$ has no torsion, it follows that a^m is injective as a map of sheaves. Hence $\mathrm{Im}(a^m|U) \cong V \otimes \mathcal{O}_U$ and we have an exact sequence on U:

$$0 \to V \otimes \mathcal{O}_U \to P^m(L)|U \to A_U \to 0 \tag{2}$$

By assumption, A_U is locally free with rank $\binom{n+m}{n} - r - 1 < n$. Hence $c_t(A_U) = 0$ for every integer t such that $n \geq t > \mathrm{rank}(A_U)$. By (2), we have $c_i(P^m(L)|U) = c_i(A_U)$ for every i, so the proof is over. □

We have also the following result.

Proposition 2

Set $R := \omega_U^{\otimes\binom{n+m}{n+1}} \otimes L^{\otimes\binom{n+m}{n}}|U$. Assume $r = \binom{n+m}{n}$ and that a^m is surjective at each point of U. Then $c_*(P^m(L)|U) = 1/c_*(R)$ in the Chow group of U.

Proof. We have $R \cong \det(P^m(L)|U)$ (use m times (1)). Since $\dim(V) = \mathrm{rank}(P^m(L)|U) + 1$ and $a^m|U$ is surjective, $\mathrm{Ker}(a^m|U)$ is a line bundle on U. This line bundle is R^* and we have the following exact sequence on U:

$$0 \to R^* \to V \otimes \mathcal{O}_U \to P^m(L)|U \to 0 \tag{3}$$

which gives the claimed relation between the Chern polynomials. □

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Here we maintain the previous notation, but from now on we have to require $\mathrm{char}(\mathbb{K}) = 0$. First of all, we show that very mild assumptions force the existence of hyperosculating points.

Theorem 5

Let $x(m) = r + 1 = \binom{n+m}{m} - 1$ and set $J := \omega_X^{\otimes m} \otimes L^{\otimes(n+1)}$. If for some integer y with $0 \leq y \leq n-2$ and a big and nef divisor H on X we have $J^{n-y} \cdot H^y > 0$ (intersection product of n divisors), then $\phi_V(X)$ must have hyperosculating points of order at most m.

Proof. We divide the proof into three steps.

Step 1) Here we assume the existence of an integral curve $D \subset X$ such that $-n-1 \leq \omega_X \cdot D < 0$ and $D \cdot L < m$. Hence the linear span $\langle \phi_V(D) \rangle$ of $\phi_V(D)$ has dimension at most $m-1$. Chosen $p \in \phi_V(D)_{reg}$ and a local coordinate corresponding to the direction of $\phi_V(D)$, we recover a linear relation about the first m partial derivatives in that direction.

Step 2) Set $R := \omega_X^{\otimes\binom{n+m}{n+1}} \otimes L^{\otimes\binom{n+m}{n}}$ and notice that a positive multiple of R is isomorphic to a positive multiple of J. By Step 1) we may assume that for every integral curve $D \subset X$ such that $-n-1 \leq \omega_X \cdot D < 0$ we have $R \cdot D \geq 0$. For every integral curve $E \subset X$ such that $\omega_X \cdot E \geq 0$ we have $R \cdot E > 0$ because L is ample. Set $\varepsilon := m/(n+1)$ and let

$$N_\varepsilon(X, L) := \{Z \in N(X)_\mathbb{R} : \omega_X \cdot Z + \varepsilon L \cdot Z \geq 0\},$$

where $N(X)_\mathbb{R}$ is the tensor product with $\mathbb{R}$ of the free group of 1-cycles on X modulo numerical equivalence ([13]). By the very definition of ε, we have $R \cdot Z \geq 0$ for every $Z \in N_\varepsilon(X, L)$. By the Cone Theorem ([13], Theorem 1.4), R is nef. Since R is a rational multiple of J, we have $R^{n-y} \cdot H^y > 0$. From [4], Lemma 2.2.7, it follows that $h^1(X, R^*) = 0$.

Step 3) Assume by contradiction that there are no hyperosculating points of order at most m. As in the proof of Proposition 2, we have an exact sequence

$$0 \to \mathcal{O}_X^{\oplus x} \to P^m(L) \to R \to 0 \tag{4}$$

where $x := \binom{n+m}{m} - 1$. By Step 2), we have $h^1(X, R^*) = 0$. Hence the exact sequence (4) splits and we obtain $P^m(L) \cong \mathcal{O}^{\oplus x} \oplus R$ with $x := \binom{n+m}{n} - 1$. Let C be the intersection of $n-1$ general members of $|L|$. Hence $(L^*)^{\oplus(n-1)}$ is the conormal module of C in X and $(\Omega_X^1|C) \otimes L)$ is an extension of $\omega_C \otimes L$ by $\mathcal{O}_C^{\oplus(n-1)}$. Hence $S^m(\Omega_X^1) \otimes L^{\otimes m}|C$ is an extension of an ample rank $\binom{n+m-1}{m} - \binom{n+m-2}{m}$ vector bundle F by the trivial rank $\binom{n+m-2}{m}$ vector bundle. By (1), there is an injection

$$j : S^m(\Omega_X^1) \otimes L^{\otimes m}|C \to P^m(L)|C \cong \mathcal{O}_C^{\oplus x} \oplus (R|C).$$

At least $\binom{n+m-2}{m} - 1$ of the trivial factors of $S^m(\Omega_X^1) \otimes L^{\otimes m}|C$ are mapped isomorphically onto some of the trivial factors of $P^m(L)|C$. Hence j induces a map $u : F \to \mathcal{O}_C^{\oplus x}$ with $\mathrm{rank}(u) \geq \mathrm{rank}(F) - 1 > 0$, in contradiction with the ampleness of F. $\square$

Remark 3. Let $C \subset \mathbb{P}^r$ be an integral non-degenerate curve and $f : X \to C$ its normalization map. Take $L := f^*(\mathcal{O}_C(1))$ and $V := f^*(H^0(\mathbb{P}^r, \mathcal{O}_{\mathbf{P}^r}(1)))$. By the Brill - Segre formula (see e.g. [10], p. 54), there are no hyperosculating points if and only if C is a rational normal curve.

Sometimes it is also possible to bound the drop of the dimension of the osculating space at a hyperosculating point. For instance, consider the following generalization of a result by Lanteri (see [11], Theorem B):

Proposition 3

Assume that X is a linear $\mathbb{P}^{n-1}$-bundle over a smooth curve. Then $\dim T(2, p, X) \geq n+1$ for every $p \in X$.

Proof. We follow an argument provided by Lanteri in [11] for the 2-dimensional case. Since

$$\dim V = \dim \mathrm{Ker}(\alpha^m(p)) + \dim \mathrm{Im}(\alpha^m(p)),$$

we have

$$\dim T(m, p, X) + \dim |V - (m+1)p| = r - 1$$

for every $m \geq 1$, and we deduce that

$$\dim T(2, p, X) = n + \mathrm{codim}(|V - 3p|, |V - 2p|).$$

Therefore it is sufficient to show that $|V - 3p| \neq |V - 2p|$ for every $p \in X$. In order to do so, let $F(p)$ be the fiber of X through p and notice that

$$|V - 3p| = 2F(p) + |V - 2F(p) - p|,$$

as it easily follows from the fact that $D^{n-1}.F(p) = 1$ for every $D \in |V|$. If $|V - 2p| = 2F(p) + |V - 2F(p) - p|$ for some point $p \in X$, then every hyperplane tangent to X at p is tangent along the whole fiber $F(p)$. In particular, the tangent space to X is constant along a positive dimensional subvariety, in contradiction with Zak's Theorem on the finiteness of the Gauss map (see for instance [22], Chapter I, Corollary 2.8).

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