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On the exact value of Jung constants of Orlicz sequence spaces

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Abstract

The main result of this paper is: $JC(\ell^{(\Phi)}) = JC(\ell^{\Phi}) = 2^{1/C_\Phi^0 - 1}$ under some condition, where $JC(\ell^{(\Phi)}), JC(\ell^{\Phi})$ are Jung constants of Orlicz sequence space ℓ^{Φ} equipped with Luxemburg and Orlicz norm, respectively.

1. Introduction

The concept of Jung constant for a normed linear space was introduced by Jung [5] in 1901 and it was termed by Grünbaum [4]:

Definition 1.1. The Jung constant of a normed linear space X, $JC(X)$ is defined as:

$$JC(X) = \sup\left\{\frac{r(A,X)}{d(A)} : A \subset X, \text{ bounded }, d(A) > 0\right\},$$

where $d(A) = \sup\{\|x-y\| : x, y \in A\}$ is the diameter of A, and

$$r(A,X) = \inf\{\sup[\|x-z\| : x \in A] : z \in X\}$$

is the absolute Chebyshev radius of A.

The definition implies $1/2 \le JC(X) \le 1$. Amir [1] obtained the following equivalent expression which is practical for calculation:

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Proposition 1.2

If $X = Y^*$, the dual space of a normed linear space Y, then

$$JC(X) = \sup\left\{\frac{r(A,X)}{d(A)} : A \subset X,\ \text{finite}\ , d(A) > 0\right\}. \tag{1}$$

Jung [5] showed that

$$JC(\ell_n^2) = \sqrt{\frac{n}{2(n+1)}},$$

where ℓ_n^2 is the Euclidean $n-$space. He also showed for an $n-$dimensional normed linear space X_n, $JC(X_n) = 1/2$ if and only if $X_n = \ell_n^\infty$. Also Bohnenblust [3] established that $JC(X_n) \leq n/(n+1)$ for any $n-$dimensional normed linear space X_n. For some infinite dimensional spaces the following results are available:

$$JC(\ell^2) = \frac{1}{\sqrt{2}}, \quad JC(L^\infty[0,1]) = JC(\ell^\infty) = \frac{1}{2}$$

and

$$JC(L^1[0,1]) = JC(\ell^1) = JC(c_0) = JC(C(T)) = 1$$

where T is a compact Hausdorff space, without isolated points. Obtaining lower and upper bounds respectively by Berdyshev [2] and Pichugov [6], the following result was deduced:

$$JC(L^p[-\pi,\pi]) = \max\left(2^{1/p-1}, 2^{-1/p}\right), 1 < p < \infty.$$

For $L^p(\Omega)$ with $\Omega = [0,1]$ or $[0,+\infty)$ and ℓ^p Ren and Chen [8], Zhang [10] obtained the similar results. In fact, "estimation of Jung constants is one of the directions of research of the geometric theory of normed spaces". However, expression for exact values of Jung constants has remained a problem. This paper is devoted to the exact value of Jung constant in a class of Orlicz sequence spaces.

Let

$$\Phi(u) = \int_0^{|u|} \varphi(t)dt \ \text{ and } \ \Psi(v) = \int_0^{|v|} \psi(s)ds$$

be a pair of complementary $N-$functions, i.e., $\varphi(t)$ is right continuous, $\varphi(0) = 0$, and $\varphi(t) \nearrow \infty$ as $t \nearrow \infty$. We call $\Phi \in \Delta_2(0)$, if there exist $u_0 > 0$ and $k > 2$ such that $\Phi(2u) \leq k\Phi(u)$ for $0 \leq u \leq u_0$. The Orlicz sequence space is defined as the set

$$\ell^\Phi = \left\{x(i) : \rho_\Phi(\lambda x) = \sum_{n=1}^{\infty} \Phi(\lambda|x(i)|) < \infty \text{ for some } \lambda > 0\right\}.$$

The Luxemburg norm and Orlicz norm are expressed as

$$\|x\|_{(\Phi)} = \inf\left\{c > 0 : \rho_\Phi\big(\frac{x}{c}\big) \leq 1\right\}$$

and

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k}\left[1 + \rho_\Phi(kx)\right],$$

respectively. In what follows, we will use Semenov and Simonenko indices of $\Phi(u)$:

$$\alpha_\Phi^0 = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi^0 = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{2}$$

$$A_\Phi^0 = \liminf_{t\to 0} \frac{t\varphi(t)}{\Phi(t)}, \qquad B_\Phi^0 = \limsup_{t\to 0} \frac{t\varphi(t)}{\Phi(t)}. \tag{3}$$

The same indices can be applied to $\Psi(v)$. The author [9] obtained

$$2\alpha_\Phi^0\beta_\Psi^0 = 1 = 2\alpha_\Psi^0\beta_\Phi^0. \tag{4}$$

Rao and Ren [7] gave the following interrelations:

$$2^{-1/A_\Phi^0} \le \alpha_\Phi^0 \le \beta_\Phi^0 \le 2^{-1/B_\Phi^0}, \tag{5}$$

If the index function $F_\Phi(t) = t\varphi(t)/\Phi(t)$ is monotonic (increase or decrease) at a right neighborhood of 0, then the limit $C_\Phi^0 = \lim\limits_{t\to 0} t\varphi(t)/\Phi(t)$ must exist, and hence

$$\alpha_\Phi^0 = \beta_\Phi^0 = 2^{-1/C_\Phi^0}. \tag{6}$$

The author [9] founded that $G_\Phi(u) = \Phi^{-1}(u)/\Phi^{-1}(2u)$ is increasing if and only if $F_\Phi(t)$ is increasing upon the corresponding interval. These relations will play important roles in our main results.

2. Main results

We need only to observe the case of $\ell^{(\Phi)}$ and ℓ^Φ being reflexive, or equivalently, $\Phi \in \triangle_2(0)\cap\nabla_2(0)$ since Rao and Ren [7] asserted that $JC(\ell^{(\Phi)}) = JC(\ell^\Phi) = 1$ if and only if ℓ^Φ is nonreflexive. [7] obtained the following results about the interpolation of Orlicz spaces:

Proposition 2.1

Let Φ be an $N-$function, $\Phi_0(u) = u^2$, and let Φ_s be the inverse of

$$\Phi_s^{-1}(u) = \left[\Phi^{-1}(u)\right]^{1-s}\left[\Phi_0^{-1}(u)\right]^s, \qquad 0 < s \le 1, u \ge 0.$$

Then

$$\max\left(\frac{1}{2\alpha_{\Phi_s}^0}, \beta_{\Phi_s}^0\right) \le \left\{JC(\ell^{(\Phi_s)}), JC(\ell^{\Phi_s})\right\} \le 2^{-s/2} \tag{7}$$

For any $N-$function Φ with $C_\Phi^0 < 2$, we produce a function M such that

$$\Phi^{-1}(u) = \left[M^{-1}(u)\right]^{1-s}\left[\Phi_0^{-1}(u)\right]^s \tag{8}$$

for some $0 < s < 1$, where $\Phi_0(u) = u^2$. It is important to show that M is an $N-$function under some conditions. We take l such that $1 < l < C_\Phi^0$ and let $s = 2(C_\Phi^0 - l)/C_\Phi^0(2-l)$, then $0 < s < 1$ and M is determined by

$$M^{-1}(u) = u^{-s/2(1-s)} \left[\Phi^{-1}(u)\right]^{1/(1-s)}. \tag{9}$$

Theorem 2.2

Let Φ be an $N-$function and $F_\Phi(t)$ be increasing, $C_\Phi^0 < 2$, Then the function M determined by (9) satisfies:

(A) $\lim_{t\to 0+} M(t)/t = 0$.

(B) M *is convex.*

Proof. In this case, for a sufficiently small $\varepsilon > 0$ there is a u_0 such that

$$u^{C_\Phi^0} \le \Phi(u) < u^{C_\Phi^0 + \varepsilon}$$

for $u \ge u_0$, or equivalently,

$$u^{1/C_\Phi^0 - \varepsilon} \le \Phi^{-1}(u) \le u^{1/C_\Phi^0}.$$

Therefore,

$$\begin{aligned} \frac{u}{M^{-1}(u)} &= \left(\frac{u^{(2-s)/2}}{\Phi^{-1}(u)}\right)^{1/(1-s)} \le \left(\frac{u^{(2-s)/2}}{u^{1/C_\Phi^0 - \varepsilon}}\right)^{1/(1-s)} \\ &= \left(u^{(C_\Phi^0 - 2)(l-1)/C_\Phi^0(l-2) + \varepsilon}\right)^{1/(1-s)} \to 0 \end{aligned}$$

as $u \to 0+$. Let $M^{-1}(u) = t$, then $u = M(t)$ and hence $\lim_{t\to 0+} M(t)/t = 0$.

To prove (B), it suffices to prove $M^{-1}(u)$ is concave. Observe that

$$\begin{aligned} M^{-1}(u) &= \frac{(\Phi^{-1})^{1/(1-s)}}{u^{s/2(1-s)}}, \\ \frac{d}{du}M^{-1} &= \frac{\frac{1}{1-s}(\Phi^{-1})^{s/(1-s)} \cdot \frac{1}{\varphi} \cdot u^{s/2(1-s)} - \frac{s}{2(1-s)} u^{3s-2/2(1-s)} (\Phi^{-1})^{1/(1-s)}}{u^{s/(1-s)}} \\ &= \frac{u(\Phi^{-1})^{s/(1-s)} - \frac{s}{2}\varphi \cdot (\Phi^{-1})^{1/(1-s)}}{u^{2-s/2(1-s)}\varphi} \cdot \frac{1}{1-s} \\ &= \frac{1 - \frac{s}{2}\frac{\varphi \cdot \Phi^{-1}}{u}}{u^{2-s/2(1-s)}\varphi} \cdot \frac{u(\Phi^{-1})^{s/(1-s)}}{1-s} \ge \frac{1 - \frac{s}{2}C_\Phi^0}{u^{2-s/2(1-s)}\varphi} \cdot \frac{u(\Phi^{-1})^{s/(1-s)}}{1-s} \\ &= \frac{\frac{2-C_\Phi^0}{2-l}}{u^{2-s/2(1-s)}\varphi} \cdot \frac{u(\Phi^{-1})^{s/(1-s)}}{1-s} > 0, \end{aligned}$$

and that

$$\begin{aligned}
\frac{d^2}{du^2}M^{-1} &= \frac{d}{du}\left[\frac{u(\Phi^{-1})^{s/(1-s)} - \frac{s}{2}\varphi\cdot(\Phi^{-1})^{1/(1-s)}}{u^{2-s/2(1-s)}\varphi}\right]\frac{1}{1-s}\\
&= \left\{\left[\frac{s}{1-s}(\Phi^{-1})^{s/(1-s)-1}\cdot\frac{1}{\varphi}\cdot u + (\Phi^{-1})^{s/(1-s)} - \frac{s}{2}\cdot\frac{1}{1-s}(\Phi^{-1})^{s/(1-s)}\cdot\frac{1}{\varphi}\cdot\varphi\right.\right.\\
&\quad\left. -\frac{s}{2}(\Phi^{-1})^{1/1-s}\cdot\frac{\varphi'}{\varphi}\right]\varphi u^{2-s/2(1-s)} - \left[\frac{\varphi'}{\varphi}u^{2-s/2(1-s)} + \frac{2-s}{2(1-s)}\varphi u^{s/2(1-s)}\right]\\
&\quad\left.\times\left(u(\Phi^{-1})^{s/(1-s)} - \frac{s}{2}\varphi\cdot(\Phi^{-1})^{1/(1-s)}\right)\right\}\cdot\frac{1}{\left(\varphi\cdot u^{2-s/2(1-s)}\right)^2}\cdot\frac{1}{1-s}\\
&= \left[\frac{s}{1-s}\left(u^2 - \Phi^{-1}\varphi\cdot u + \frac{2-s}{4}(\Phi^{-1})^2\varphi^2\right) - \frac{\varphi'}{\varphi}\Phi^{-1}\cdot u^2\right]\\
&\quad\times\frac{(\Phi^{-1})^{s/(1-s)-1}u^{2-s/2(1-s)-1}}{(1-s)\left(\varphi\cdot u^{2-s/2(1-s)}\right)^2}\\
&= \left[\frac{2(C_\Phi^0-l)}{l(2-C_\Phi^0)}\left(1-\frac{\Phi^{-1}\varphi}{u} + \frac{C_\Phi^0+l-C_\Phi^0 l}{2C_\Phi^0(2-l)}\left(\frac{\Phi^{-1}\varphi}{u}\right)^2\right) - \frac{\varphi'\Phi^{-1}}{\varphi}\right]\\
&\quad\times\frac{(\Phi^{-1})^{2s-1/(1-s)}}{(1-s)\varphi^2 u^{s/2(1-s)}}.
\end{aligned}$$

Let $\Phi^{-1}(u) = t$, then $u = \Phi(t)$. It remains to check that

$$f(t) := \frac{2(C_\Phi^0-l)}{l(2-C_\Phi^0)}\left(1-\frac{t\varphi}{\Phi} + \frac{C_\Phi^0+l-C_\Phi^0 l}{2C_\Phi^0(2-l)}\left(\frac{t\varphi}{\Phi}\right)^2\right) - \frac{t\varphi'}{\varphi} < 0. \tag{10}$$

Since $F_\Phi(t) = t\varphi/\Phi(t)$ is increasing from C_Φ^0, we have

$$F'_\Phi(t) = \frac{(t\varphi'+\varphi)\Phi - t\varphi^2}{\Phi^2} = \frac{\varphi\left(\frac{t\varphi'}{\varphi} + 1 - \frac{t\varphi}{\Phi}\right)}{\Phi} \geq 0.$$

Therefore,

$$\frac{t\varphi'}{\varphi} + 1 - \frac{t\varphi}{\Phi} \geq 0,$$

or

$$\frac{t\varphi'}{\varphi} \geq \frac{t\varphi}{\Phi} - 1.$$

Thus,

$$f(t) \leq \frac{2(C_\Phi^0-l)}{l(2-C_\Phi^0)}\left(1-\frac{t\varphi}{\Phi} + \frac{C_\Phi^0+l-C_\Phi^0 l}{2C_\Phi^0(2-l)}\left(\frac{t\varphi}{\Phi}\right)^2\right) - \left(\frac{t\varphi}{\Phi} - 1\right). \tag{11}$$

Since the function

$$\begin{aligned}
g(x) :&= \frac{2(C_\Phi^0-l)}{l(2-C_\Phi^0)}\left(1 - x + \frac{C_\Phi^0+l-C_\Phi^0 l}{2C_\Phi^0(2-l)}x^2\right) - (x-1)\\
&= \frac{(C_\Phi^0-l)}{l(2-C_\Phi^0)}\left[\frac{C_\Phi^0+l-C_\Phi^0 l}{C_\Phi^0(2-l)}x^2 - \frac{C_\Phi^0(2-l)}{C_\Phi^0-l}(x-1)\right]
\end{aligned}$$

is decreasing for

$$x \leq [C^0_\Phi(l-2)]^2/[2(C^0_\Phi + l - C^0_\Phi l)(C^0_\Phi - l)].$$

If $C^0_\Phi < 2$, then

$$C^0_\Phi \leq [C^0_\Phi(l-2)]^2/[2(C^0_\Phi + l - C^0_\Phi l)(C^0_\Phi - l)]$$

for a sufficiently small $l > 1$ under the condition $C^0_\Phi < 2$. It follows that

$$f(t) \leq \frac{(C^0_\Phi - l)}{l(2 - C^0_\Phi)} \left[\frac{C^0_\Phi + l - C^0_\Phi l}{C^0_\Phi (2-l)} (C^0_\Phi)^2 - \frac{C^0_\Phi(2-l)}{C^0_\Phi - l}(C^0_\Phi - 1) \right] \tag{12}$$

since $F_\Phi(t) \geq C^0_\Phi$.

We only need to check that

$$h(l) := \frac{2(C^0_\Phi - l)}{l(2 - C^0_\Phi)} \left(1 - C^0_\Phi + \frac{C^0_\Phi + l - C^0_\Phi l}{2C^0_\Phi(2-l)} \left(C^0_\Phi\right)^2 \right) - (C^0_\Phi - 1) < 0,$$

or equivalently,

$$h(l) = \frac{C^0_\Phi \left[-(C^0_\Phi)^2 l + 4C^0_\Phi l + (C^0_\Phi)^2 - 4C^0_\Phi - 4l + 4 \right]}{l(2-l)(2 - C^0_\Phi)} < 0.$$

Let $l \to 1+$, then $h(l) \to 0$. On the other hand, since

$$h'(l) = \frac{C^0_\Phi}{2 - C^0_\Phi} \cdot \frac{-\left[l(C^0_\Phi - 2) + 2 \right]^2 - 8C^0_\Phi(l-1) - 4}{(2l - l^2)^2} < 0,$$

we see that $h(l)$ is decreasing on $(1, C^0_\Phi)$, and hence we deduce that $h(l) < h(1+) = 0$ on its domain. Therefore, we proved M is convex. The proof is finished. □

Remark 2.3. If $C^0_\Phi > 2$, then the parameter l can be taken such that $C^0_\Phi < l < +\infty$ and $l \to +\infty$. But (12) of Theorem 2.2 is no longer substitutable since $g(x)$ is increasing for $x > C^0_\Phi$. Thus, the proof of convexity of M have to be restricted to $1 < C_\Phi < 2$.

Theorem 2.4

Let Φ be an $N-$function. $F_\Phi(t) = t\varphi(t)/\Phi(t)$ is increasing, $1 \leq C^0_\Phi < 2$, *then*

$$JC(l^{(\Phi)}) = JC(l^\Phi) = 2^{1/C^0_\Phi - 1}. \tag{13}$$

Proof. It follows from (7) and (8) that

$$\max\left(\frac{1}{2\alpha^0_\Phi}, \beta^0_\Phi \right) \leq J(l^{(\Phi)}) \leq 2^{-s/2} \tag{14}$$

when $F_\Phi(t)$ is increasing and $1 < C^0_\Phi < 2$. Since $F_\Phi(t)$ is increasing, $\frac{1}{2\alpha^0_\Phi} = 2^{1/C^0_\Phi - 1}$ by (6). On the other hand, in (14) let $l \to 1+$, then $(2-s)/2 \to 1/C^0_\Phi$. Therefore, (13) holds. It is obvious to see that (13) also holds for $C^0_\Phi = 1$ since the spaces generated by Φ is nonreflexive. □

Corollary 2.5

Let Φ be an $N-$function. $F_\Phi(t) = t\varphi(t)/\Phi(t)$ is increasing, φ is concave, then

$$JC(\ell^{(\Phi)}) = JC(\ell^\Phi) = 2^{1/C_\Phi^0 - 1}. \tag{15}$$

Proof. When φ is concave,

$$\varphi(t) = \int_0^t \varphi'(s)ds + \varphi(0) = \int_0^t \varphi'(s)ds \geq t\varphi'(t).$$

Therefore,

$$[t\varphi(t) - 2\Phi(t)]' = t\varphi' - \varphi \leq 0,$$

and hence $t\varphi(t) - 2\Phi(t) \leq 0$, in other words, $F_\Phi(t) \leq 2$ which means $C_\Phi^0 \leq 2$. If $C_\Phi^0 < 2$, then (15) holds by Theorem 2.2. If $C_\Phi^0 = 2$, then $t\varphi(t)/\Phi(t) \geq 2$ since $F_\Phi(t)$ is increasing form 0, and hence, $t\varphi(t)/\Phi(t) = 2$ at a neighborhood of 0. This means that $\Phi(t) = at^2 (a > 0)$, which generates the Hilbert space ℓ^2, so (15) holds by the well known result. □

Example 2.6: The $N-$function $\Phi(u) = 2|u|^p + |u|^{2p}, \quad 1 < p < 2$ satisfies

$$F_\Phi(t) = \frac{t\Phi'(t)}{\Phi(t)} = 2p\left(\frac{t^p + 1}{t^p + 2}\right) \geq p$$

and $C_\Phi^0 = p$. Therefore, we have

$$JC(\ell^{(\Phi)}) = JC(\ell^\Phi) = 2^{1/C_\Phi^0 - 1} = 2^{1/p - 1}. \tag{16}$$

Let $0 < s \leq 1$, then we can produce Φ_s by

$$\Phi_s^{-1}(u) = (\sqrt{u+1} - 1)^{(1-s)/p} u^{s/2}.$$

Consequently,

$$\alpha_{\Phi_s}^0 = \beta_{\Phi_s}^0 = \lim_{u\to 0} \frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)} = \lim_{u\to 0}\left(\frac{\sqrt{u+1}-1}{\sqrt{2u+1}-1}\right)^{(1-s)/p} \cdot \left(\frac{1}{2}\right)^{s/2} = \left(\frac{1}{2}\right)^{(1-s)/p + s/2}.$$

Since the author [9] proved that the function $G_{\Phi_s}(u) = \frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)}$ is increasing on $(0, +\infty)$ if and only if $F_{\Phi_s}(t)$ is increasing on $(0, +\infty)$, we deduce that $F_{\Phi_s}(t)$ is increasing although it is impossible to express. Thus,

$$C_{\Phi_s}^0 = \frac{1}{\frac{1-s}{p} + \frac{s}{2}} \in (1, 2)$$

and hence we obtain the exact value:

$$JC(\ell^{(\Phi)}) = JC(\ell^\Phi) = 2^{1/C_{\Phi_s}^0 - 1} = 2^{(1-s)/p + s/2 - 1}. \tag{17}$$

EXAMPLE 2.7: Let $\Phi_p(u) = e^{|u|^p} - 1, 1 < p < 2$. Rao and Ren ([7], pp. 143) obtained

$$2^{1/p-1} \le JC(\ell^{(\Phi_p)}) < 1.$$

Since F_{Φ_p} is increasing on $(0,\infty)$ and $C^0_{\Phi_p} = p$. Therefore, we have the exact value

$$JC(\ell^{(\Phi_p)}) = JC(\ell^{\Phi_p}) = 2^{1/C^0_{\Phi_p}-1} = 2^{1/p-1}. \tag{18}$$

Moreover, let Φ_s be a function defined as the inverse of

$$\Phi_s^{-1}(u) = [\ln(1+u)]^{(1-s)/p}\, u^{s/2}, 0 < s \le 1. \tag{19}$$

Then Φ_s is just the interpolation of $\Phi_1(u) = e^{|u|^p} - 1$ and $\Phi_0(u) = u^2$. Therefore, analogous to Example 2.6, we obtain

$$JC(\ell^{(\Phi_s)}) = JC(\ell^{\Phi_s}) = 2^{1/C^0_{\Phi_s}-1} = 2^{(1-s)/p+s/2-1}. \tag{20}$$

Observe that [7] (pp. 144) gave the estimation:

$$2^{(1-s)/p+s/2-1} \le \left\{ JC(\ell^{(\Phi_s)}), JC(\ell^{\Phi_s}) \right\} \le 2^{-s/2}.$$

Particularly, for $s = 1/2$, we obtain

$$\Phi^{-1}_{1/2}(u) = [\ln(1+u)]^{1/2p}\, u^{1/4}.$$

Therefore,

$$JC(\ell^{(\Phi_{1/2})}) = JC(\ell^{\Phi_{1/2}}) = 2^{1/C^0_{\Phi_{1/2}}-1} = 2^{1/2p-3/4}. \tag{21}$$

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