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The space $bv(p)$, its β–dual and matrix transformations

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Abstract

Let $p = (p_k)_{k=0}^\infty$ be a sequence with $p_k > 0$ for all k. We consider the space $bv(p) = \{x \in \omega : \sum_{k=0}^\infty |x_k - x_{k-1}|^{p_k} < \infty\}$, study its β–dual and characterize some matrix transformations on $bv(p)$ which yield the results in [16] and [13] as special cases.

1. Introduction

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^\infty$. Let ϕ, ℓ_∞ and c_0 denote the set of all finite, bounded and null sequences. We write

$$\ell_p = \left\{ x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty \right\}$$

for $0 < p < \infty$.

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By e and $e^{(n)}$ $(n \in \mathbb{N}_0)$, we denote the sequences with $e_k = 1$ $(k = 0, 1, \dots)$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n–section.

Let $p = (p_k)_{k=0}^{\infty}$ be a sequence of strictly positive reals throughout. The sets

$$\ell(p) = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty \right\},$$
$$\ell_\infty(p) = \left\{ x \in \omega : \sup_k |x_k|^{p_k} < \infty \right\},$$
$$c_0(p) = \left\{ x \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0 \right\} \text{ and}$$
$$c(p) = \{ x \in \omega : x - le \in c_0(p) \text{ for some } l \in \mathbb{C} \},$$

were first introduced and studied by Nakano, [14], Simons [15] and Maddox [4]. If $p_k = p$ $(k = 0, 1, \dots)$ for constant $p > 0$ then these sets reduce to l_p, ℓ_∞, c_0 and c.

Given any sequence x, we write Δx for the sequence with $\Delta x_k = x_k - x_{k-1}$ for $k = 0, 1, \dots$, and use the convention that any term with a negative subscript is equal to zero. We consider the set

$$bv(p) = \{ x \in \omega : \Delta x \in \ell(p) \} = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k - x_{k-1}|^{p_k} < \infty \right\};$$

if $p_k = p > 1$ for all $k = 0, 1, \dots$ where p is a constant, then $bv(p)$ reduces to the set bv^p studied in [13], and $bv(e) = bv$, the well–known set of sequences of bounded variation.

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers. We write

$$xy = (x_k y_k)_{k=0}^{\infty}, \quad x^{-1} * Y = \{ a \in \omega : ax \in Y \}$$

and

$$M(X, Y) = \cap_{x \in X} x^{-1} * Y = \{ a \in \omega : ax \in Y \text{ for all } x \in X \}$$

for the *multiplier space of* X *and* Y. In the special case of $Y = cs$, $X^\beta = M(X, cs)$ is the β*–dual of* X. By A_n we denote the sequence in the n–th row of A, and we write $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$ $(n = 0, 1, \dots)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$, provided $A_n \in x^\beta$ for all n. Furthermore (X, Y) denotes the class of all matrices that map X into Y, that is $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n and $A(x) \in Y$ for all $x \in X$.

An FK *space* is a complete linear metric space with the property that convergence implies coordinatewise convergence; a BK *space* is a normed FK space. An FK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is $x = \lim_{n\to\infty} x^{[n]}$. The space $\ell(p)$ is an FK space with AK if and only if $p \in \ell_\infty$, with its metric given by the paranorm $g(x) = (\sum_{k=0}^{\infty} |x_k|^{p_k})^{1/M}$ where $M = \max\{1, \sup_k p_k\}$ ([14], [5] and [6]). Thus if $p \in \ell_\infty$ then, by [17, Theorem 4.3.12, p. 63], $bv(p)$ is an FK space with $g(x) = (\sum_{k=0}^{\infty} |x_k - x_{k-1}|^{p_k})^{1/M}$. Further results on the topological structures of the spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$ can be found in [1]; they are not needed here.

In this paper, we study the β–dual of the set $bv(p)$, and determine $(bv(p))^\beta$ when $p \in \ell_\infty$, thus extending the result given in [13]. Furthermore, we characterize the classes

$(bv(p), \ell(s))$, $(bv(p), \ell_\infty(s))$, $(bv(p), c_0(s))$, $(bv(p), c(s))$ for bounded positive sequences $s = (s_k)_{k=0}^\infty$, $(bv(p), \ell_1)$, $(bv(p), \ell_\infty)$, $(bv(p), c_0)$ and $(bv(p), c)$ and obtain the results in [16] and [13] as special cases.

2. The β–dual of $bv(p)$

In this section, we study the β–dual of $bv(p)$ and some special cases.

2.1. The case $p_k > 1$.

Throughout this subsection, let p be a sequence with $p_k > 1$ and $q_k = p_k/(p_k - 1)$ for $k = 0, 1, \ldots$. We write $\ell_1^+ = \{x \in \ell_1 : x_k \geq 0 \text{ for all } k\}$. If $a \in cs$, then we define the sequence R by $R_k = \sum_{j=k}^\infty a_j$ for $k = 0, 1, \ldots$.

We need the following results.

Lemma 2.1

We put

$$M_1(p) = \bigcup_{N \in \mathbb{N} \setminus \{1\}} \left\{ a \in \omega : \sum_{k=0}^\infty |R_k|^{q_k} N^{-q_k} < \infty \right\},$$

$$M_2(p) = \bigcap_{v \in \ell_1^+} \left\{ a \in \omega : \sum_{k=0}^\infty a_k \sum_{j=0}^k v_j^{1/p_j} \text{ converges} \right\}$$

and $M(p) = M_1(p) \cap M_2(p)$. Then we have $(bv(p))^\beta = M(p)$.

Proof. First we assume $a \in M(p)$. Since $e^{(0)} \in \ell_1^+$, $a \in M_2(p)$ implies $a \in cs$, and so the sequence R is defined. Abel's summation by parts yields

$$\sum_{k=0}^n a_k x_k = \sum_{k=0}^{n+1} R_k \Delta x_k - R_{n+1} x_{n+1} \quad (n = 0, 1, \ldots) \text{ for all } x. \tag{2.1}$$

By [7, Theorem 1], $a \in M_1(p)$ implies $R \in (\ell(p))^\beta$. Since $x \in bv(p)$ if and only if $\Delta x \in \ell(p)$, we conclude

$$R\Delta x \in cs \text{ for all } x \in bv(p). \tag{2.2}$$

Let $x \in bv(p)$ be given. We put $v_k = |\Delta x_k|^{p_k}$ for $k = 0, 1, \ldots$. Then $v \in \ell_1^+$ and

$$|R_{n+1} x_{n+1}| \leq |R_{n+1}| \sum_{k=0}^{n+1} |x_k - x_{k-1}| \leq |R_{n+1}| \sum_{k=0}^{n+1} v_k^{1/p_k} \text{ for all } n.$$

By [10, Corollary 1], $a \in M_2(p)$ implies

$$R_{n+1} \sum_{k=0}^{n+1} v_k^{1/p_k} \to 0 \quad (n \to \infty), \tag{2.3}$$

hence

$$Rx \in c_0. \tag{2.4}$$

Now (2.1), (2.2) and (2.4) imply $ax \in cs$ for all $x \in bv(p)$, that is $a \in (bv(p))^\beta$.

Conversely, we assume $a \in (bv(p))^\beta$. Then $ax \in cs$ for all $x \in bv(p)$, and $e \in bv(p)$ implies $a \in cs$. Therefore the sequence R is defined. Let $v \in \ell_1^+$ be given. We define the sequence x by $x_k = \sum_{j=0}^k v^{1/p_j}$ for $k = 0, 1, \ldots$. Then we obtain $\sum_{k=0}^\infty |\Delta x_k|^{p_k} = \sum_{k=0}^\infty |v_k| < \infty$, that is $x \in bv(p)$, and consequently $ax \in cs$, that is $a \in M_2(p)$. Now [10, Corollary 1] implies (2.3), and thus (2.4) holds for all $x \in bv(p)$. Finally, from (2.1), we obtain (2.2). Since $x \in bv(p)$ if and only if $\Delta x \in \ell(p)$, this implies $R \in (\ell(p))^\beta$ and $(\ell(p))^\beta = M_1(p)$ by [7, Theorem 1]. □

Lemma 2.2

We have $a \in (bv(p))^\beta$ if and only if $R \in (\ell(p))^\beta \cap M(bv(p), c_0)$.

Proof. If $R \in (\ell(p))^\beta \cap M(bv(p), c_0)$ then $a \in (bv(p))^\beta$ by (2.1).

Conversely, if $a \in (bv(p))^\beta$ then $R \in (\ell(p))^\beta$ and $\sum_{k=0}^\infty a_k \sum_{j=0}^k v_j^{1/p_j}$ converges for all $v \in \ell_1^+$ by Lemma 2.1. This implies $Rx \in c_0$ for all $x \in bv(p)$, as in the first part of the proof of Lemma 2.1. □

As an immediate consequence of (2.1) and Lemma 2.2, we obtain

Corollary 2.1

If $a \in (bv(p))^\beta$ then we have

$$\sum_{k=0}^\infty a_k x_k = \sum_{k=0}^\infty R_k \Delta x_k \text{ for all } x \in bv(p). \tag{2.5}$$

Lemma 2.3

We put

$$M_3(p) = \bigcup_{N \in \mathbb{N} \setminus \{1\}} \left\{ a \in \omega : \lim_{n \to \infty} \sum_{k=0}^n |a_n|^{q_k} N^{-q_k/p_k} = 0 \right\}$$

and

$$M_4(p) = \bigcup_{N \in \mathbb{N} \setminus \{1\}} \left\{ a \in \omega : \sup_n \sum_{k=0}^n |a_n|^{q_k} N^{-q_k/p_k} < \infty \right\}.$$

(a) Then we have $M_3(p) \cap c_0 \subset M(bv(p), c_0)$.
(b) If $h = inf_k p_k > 1$ then we have $M(bv(p), c_0) \subset M_4(p) \cap c_0$.

Proof. (a) First we assume $a \in M_3(p) \cap c_0$. Let $\varepsilon > 0$ and $x \in bv(p)$ be given. We put $y = \Delta x \in \ell(p)$ and choose $m_0 \in \mathbb{N}$ such that

$$\sum_{k=m}^\infty |y_k|^{p_k} < \varepsilon \text{ for all } m \geq m_0. \tag{2.6}$$

Since $a \in c_0$ and $a \in M_3(p)$, we can choose $n_0 \in \mathbb{N}$ and $N \in \mathbb{N} \setminus \{1\}$ such that

$$\sum_{k=0}^{m_0} |a_n||y_k| < \varepsilon \text{ for all } n \geq n_0 \tag{2.7}$$

and

$$\sum_{k=0}^{n} |a_n|^{q_k} N^{-q_k/p_k} < \varepsilon \text{ for all } n \geq n_0. \tag{2.8}$$

Let $n \geq n_0$ be given. By (2.7), [3, (2)], (2.6) and (2.8), we have

$$\begin{aligned} |a_n x_n| &\leq \sum_{k=0}^{m_0} |a_n y_k| + \sum_{k=m_0+1}^{n} |a_n y_k| \\ &< \varepsilon + N \left(\sum_{k=m_0+1}^{n} |a_n|^{q_k} N^{-q_k} + \sum_{k=m_0+1}^{n} |y_k|^{p_k} \right) \\ &< \varepsilon(1+N) + \sum_{k=m_0+1}^{n} |a_n|^{q_k} N^{-q_k/p_k} < \varepsilon(2+N), \end{aligned}$$

that is $a \in M(bv(p), c_0)$.

(b) Now let $\inf_k p_k = h > 1$. We assume $a \in M(bv(p), c_0)$. First $e \in bv(p)$ implies $a \in c_0$. We assume $a \notin M_4(p)$. Then

$$\sup_n \sum_{k=0}^{n} |a_n|^{q_k} N^{-q_k/p_k} = \infty \text{ for all } N \in \mathbb{N} \setminus \{1\}. \tag{2.9}$$

We put $n(0) = -1$. First, by (2.9), we can choose $n(1) \in \mathbb{N}$ such that

$$\sum_{k=n(0)+1}^{n(1)} |a_{n(1)}|^{q_k} 2^{-q_k/p_k} > 3. \tag{2.10}$$

Since $a \in c_0$, we can choose $m(1) > n(1)$ such that

$$\sum_{k=0}^{n(1)} |a_m|^{q_k} 3^{-q_k/p_k} < 1 \text{ for all } m \geq m(1) \tag{2.11}$$

and

$$|a_m| < \frac{1}{3} |a_{n(1)}| \text{ for all } m \geq m(1) \text{ (note } |a_{n(1)}| > 0 \text{ by (2.10))}. \tag{2.12}$$

Now, by (2.9), we choose $n(2) > m(1)$ such that

$$\sum_{k=n(0)+1}^{n(2)} |a_{n(2)}|^{q_k} 3^{-q_k/p_k} > 4. \tag{2.13}$$

Then, by (2.13) and (2.11)

$$\sum_{k=n(1)+1}^{n(2)} |a_{n(2)}|^{q_k} 3^{-q_k/p_k} = \sum_{k=n(0)+1}^{n(2)} |a_{n(2)}|^{q_k} 3^{-q_k/p_k} - \sum_{k=n(0)+1}^{n(1)} |a_{n(2)}|^{q_k} 3^{-q_k/p_k} > 3.$$

Continuing in this way, we can define an increasing sequence $(n(j))_{j=0}^{\infty}$ of integers such that

$$M_j = \sum_{k=n(j-1)+1}^{n(j)} |a_{n(j)}|^{q_k} (j+1)^{-q_k/p_k} > j+1 \quad (j = 1, 2, \dots) \tag{2.14}$$

and

$$|a_{n(j)}| < \frac{1}{3}|a_{n(j-1)}| \quad (j = 1, 2, \dots). \tag{2.15}$$

We put $\alpha = h - 1 > 0$ and define the sequences y and x by

$$y_k = \operatorname{sgn}(a_{n(j)})\,|a_{n(j)}|^{q_k - 1}(j+1)^{-q_k/p_k}\,M_j^{-1} \quad \text{for } n(j-1)+1 \le k \le n(j) \quad (j = 1, 2, \dots)$$

and $x_k = \sum_{j=0}^{k} y_k$ for $k = 0, 1, \dots$. Then, since $M_j > j+1$ for all j, $p_k \ge 1 + \alpha$ for all k, $\alpha > 0$ and $q_k/p_k - q_k = -1$ for all k, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} |y_k|^{p_k} &= \sum_{j=1}^{\infty} \sum_{k=n(j-1)+1}^{n(j)} |a_{n(j)}|^{q_k}(j+1)^{-q_k} M_j^{-p_k} \\
&\le \sum_{j=0}^{\infty} \frac{1}{(j+1)^{\alpha}}\,M_j^{-1} \sum_{k=n(j-1)+1}^{n(j)} |a_{n(j)}|^{q_k}(j+1)^{-q_k/p_k}\,(j+1)^{-q_k+q_k/p_k} \\
&= \sum_{j=0}^{\infty} \frac{1}{(j+1)^{1+\alpha}} M_j^{-1} M_j = \sum_{j=0}^{\infty} \frac{1}{(j+1)^{1+\alpha}} < \infty,
\end{aligned}$$

that is $y \in \ell(p)$, and so $x \in bv(p)$. But, on the other hand,

$$\begin{aligned}
|a_{n(j)} x_{n(j)}| = \left| a_{n(j)} \sum_{k=0}^{n(j)} y_k \right| &\ge \sum_{k=n(j-1)+1}^{n(j)} |a_{n(j)}|^{q_k}\,(j+1)^{-q_k/p_k}\,M_j^{-1} \\
&\quad - |a_{n(j)}| \sum_{l=1}^{j-1} \sum_{k=n(l-1)+1}^{n(l)} |a_{n(l)}|^{q_k - 1}\,(l+1)^{-q_k/p_k}\,M_l^{-1} \\
&\ge 1 - \sum_{l=1}^{j-1} \frac{1}{3^l}\,M_l^{-1} \sum_{k=n(l-1)+1}^{n(l)} |a_{n(l)}|^{q_k}(l+1)^{-q_k/p_k} \\
&\ge 1 - \sum_{l=1}^{j-1} 3^{-l} \ge 1 - \frac{1}{2} = \frac{1}{2} \text{ for all } j = 1, 2, \dots,
\end{aligned}$$

that is $ax \notin c_0$. □

Remark 1. In the case of Lemma 2.3 (b), we may have

$$M_4(p) \cap c_0 \ne M(bv(p), c_0).$$

Proof. We put $p_k = k + 2$, $x_k = (k+1)/2$ and $a_k = 1/(k+1)$ for $k = 0, 1, \dots$ Then it follows that

$$\sum_{k=0}^{\infty} |x_k - x_{k-1}|^{p_k} = \sum_{k=0}^{\infty} 2^{-(k+2)} < \infty,$$

that is $x \in bv(p)$,

$$\sum_{k=0}^{n} |a_n|^{q_k}\,2^{-q_k/p_k} = \sum_{k=0}^{n} (n+1)^{-(k+2)/(k+1)}\,2^{-1/(k+1)} \le \sum_{k=0}^{n} \frac{1}{n+1} = 1$$

for all n, and trivially $a \in c_0$, thus $a \in M_4(p) \cap c_0$, but $a_n x_n = 1/2$ for all n, that is $a \notin M(bv(p), c_0)$. $\square$

Now we give the β–dual of the set $bv(p)$ when the sequence p is bounded.

Theorem 2.1

Let $p \in \ell_\infty$ and $p_k > 1$ for all $k = 0, 1, \ldots$. Then $a \in (bv(p))^\beta$ if and only if for some $N \in \mathbb{N} \setminus \{1\}$

$$\sum_{k=0}^{\infty} |R_k|^{q_k} N^{-q_k} < \infty \tag{2.16}$$

and

$$\sup_n \sum_{k=0}^{n} |R_n|^{q_k} N^{-q_k} < \infty. \tag{2.17}$$

Proof. We have $a \in (bv(p))^\beta$ by Lemma 2.2 if and only if $R \in (\ell(p))^\beta$, that is if and only if (2.16) holds by [7, Theorem 1], and $R \in M(bv(p), c_0)$. We show that $R \in M(bv(p), c_0)$ if and only if (2.17) holds. To do this, we define the matrix $C = (c_{nk})_{n,k=0}^\infty$ by $c_{nk} = R_n$ for $0 \le k \le n$ and $c_{nk} = 0$ for $k > n$ $(n = 0, 1, \ldots)$. Since $x \in bv(p)$ if and only if $y = \Delta x \in \ell(p)$, we have $R \in M(bv(p), c_0)$ if and only if $C \in (\ell(p), c_0)$ which by [3, Corollary] is the case if and only if

$$\lim_{n\to\infty} c_{nk} = \lim_{n\to\infty} R_n = 0,$$

which trivially holds since $R_n = \sum_{k=n}^\infty a_n$, and

$$\sup_n \sum_{k=0}^{\infty} |c_{nk}|^{q_k} N^{-q_k/p_k} = \sup_n \sum_{k=0}^{n} |R_n|^{q_k} N^{-q_k/p_k} < \infty$$

for some $N \in \mathbb{N} \setminus \{1\}$, and the last condition obviously is equivalent with (2.17). $\square$

There is an alternative proof of Theorem 2.1 in which Lemma 2.2 is not needed.

Remark 2. When $p \in \ell_\infty$ then $\ell(p)$ is an FK space with AK with respect to the paranorm g defined by $g(x) = (\sum_{k=0}^\infty |x_k|^{p_k})^{1/H}$ where $H = \sup_k p_k$, and we would conclude that $a \in (bv(p))^\beta$ if and only if $R \in \ell(p) \cap M(bv(p), c)$ by [12, Theorem 2.5]. Furthermore, as in the proof of Theorem 2.1, we would be able to show $R \in (bv(p), c)$ if and only if condition (2.17) holds.

Now we consider a few special cases. Part (b) of the following remark is [13, Theorem 2.1]

Remark 3. Let $p \in \ell_\infty$ and $p_k > 1$ for all k.
(a) If $h = \inf_k p_k > 1$ and $M(bv(p)) = \{a \in \omega : \sup_n \sum_{k=0}^n |a_n|^{q_k} < \infty\}$ then

$$a \in (bv(p))^\beta \text{ if and only if } R \in \ell(q) \cap M(bv(p)). \tag{2.18}$$

Moreover

$$a \in M(bv(p), c_0) \text{ if and only if } \sup_n \sum_{k=0}^{n} |a_n|^{q_k} < \infty. \tag{2.19}$$

(b) Let $p_k = p > 1$ for all k and $M(bv^p) = ((n+1)^{1/q})^{-1} * \ell_\infty$. Then we have

$$a \in bv^p \text{ if and only if } R \in \ell_q \cap M(bv^p). \tag{2.20}$$

Furthermore neither $\ell_q \subset M(bv^p)$ nor $M(bv^p) \subset \ell_q$ (cf. [13, Remark 1]).

Proof. (a) Let $h > 1$ and $H = \sup_k p_k < \infty$. Then it follows that $(\ell(p))^\beta = \ell(q)$ by [7, Theorem 4]. We show condition (2.19). First we assume $a \in M(bv(p), c_0)$. Then there is $N \in \mathbb{N} \setminus \{1\}$ such that

$$\sup_n \sum_{k=0}^{n} |a_n|^{q_k} N^{-q_k} < \infty, \tag{2.21}$$

as we have seen in the proof of Theorem 2.1, and this implies

$$\sup_n \sum_{k=0}^{n} |a_n|^{q_k} \leq N^{H/(h-1)} \sup_n \sum_{k=0}^{n} |a_k|^{q_k} N^{-q_k} < \infty.$$

Conversely, we assume

$$\sup_n \sum_{k=0}^{n} |a_n|^{q_k} < \infty. \tag{2.22}$$

Then obviously condition (2.21) is satisfied for all $N \in \mathbb{N} \setminus \{1\}$. Furthermore it follows that $a \in c_0$. For otherwise, if $a \notin c_0$ there would be a real c with $0 < c < 1$ and a subsequence $(a_{n(j)})_{j=0}^\infty$ of the sequence a such that $|a_{n(j)}| \geq c$ for all j, and so $\sum_{k=0}^{n(j)} |a_{n(j)}|^{q_k} \geq \sum_{k=0}^{n(j)} c^{q_k} \geq c^{H/(h-1)}(n(j)+1)$ for all j, contrary to the assumption that (2.22) is satisfied. Finally, (2.22) and $a \in c_0$ together imply $a \in M(bv(p), c_0)$ by [3, Corollary].

Now (2.18) is clear.

(b) The condition in (2.20) is an immediate consequence of Part (a). □

Concerning the proof of Remark 3 (a) we note.

Remark 4. If $h = 1$ then condition (2.21) does not imply $a \in c$.

Proof. We choose $p_k = 1 + 1/(k+1)$ $(k = 0, 1, \dots)$ and $a_n = (-1)^n$ $(n = 0, 1, \dots)$. Then $a \notin c$, but $\sup_n \sum_{k=0}^n |a_n|^{q_k} 2^{-q_k} = \sup_n \sum_{k=0}^n 2^{-(k+1)} < \infty$. □

2.2. The case $p_k \leq 1$.

Now we determine the β–dual of $bv(p)$ when $p_k \leq 1$ for all $k = 0, 1, \dots$.

Lemma 2.4

Let $p_k \leq 1$ for all k. Then we have $M(bv(p), c) = c$.

Proof. First, we assume $a \in M(bv(p), c)$. Since $e \in bv(p)$, this implies $a \in c$.

Conversely we assume $a \in c$. This implies $a \in \ell_\infty$, hence $\sup_{n,k} |a_n|^{p_k} < \infty$, since $p_k \leq 1$ for all k. We define the matrix $C = (c_{nk})_{n,k=0}^\infty$ by $c_{nk} = a_n$ for $0 \leq k \leq n$ and $c_{nk} = 0$ for $k > n$ $(n = 0, 1, \dots)$. From $a \in c$ and $\sup_{n,k} |c_{nk}|^{p_k} < \infty$, we conclude $C \in (\ell(p), c)$ by [3, Corollary]. Now, since $x \in bv(p)$ if and only if $y = \Delta x \in \ell(p)$, and $a_n x_n = a_n \sum_{k=0}^n y_k$, the fact that $C \in (\ell(p), c)$ implies $a \in M(bv(p), c)$. □

Theorem 2.2

Let $p_k \le 1$ for all k. Then we have $a \in (bv(p))^\beta$ if and only if

$$\sup_n \left| \sum_{k=n}^{\infty} a_k \right|^{p_n} < \infty.$$

Proof. By [3, Theorem 1 (ii)], [12, Theorem 2.5] and [10, Corollary 1], we have $a \in (bv(p))^\beta$, if and only if $R \in (\ell(p))^\beta = \ell_\infty(p)$ and $R \in M(bv(p), c) = c$, the last condition being redundant, since $R_n = \sum_{k=n}^{\infty} a_k$ for $n = 0, 1, \dots$. □

Remark 5. If $p = e$, then by Theorem 2.2 $bv^\beta = \{a \in \omega : \sup_n |\sum_{k=n}^{\infty} a_k| < \infty\}$, and so obviously $bv^\beta = cs$, a well–known result (cf. [17, Theorem 7.3.5 (iii)]).

3. Matrix transformations on $bv(p)$

In this section, we characterize some matrix transformations on $bv(p)$ and consider some special cases.

Theorem 3.1

Let $p, s \in \ell_\infty$. We assume $p_k \le 1$ and $s_k \ge 1$ in **(1.)**, *and $s_k \le 1$ in* **(3.)** *and* **(4.)** *below. Then the conditions for $A \in (bv(p), Y)$ for $Y = \ell(s), \ell_\infty(s), c_0(s), c(s)$ can be read from the table*

To / From	$\ell(s)$	$\ell_\infty(s)$	$c_0(s)$	$c(s)$
$bv(p)$	(1.)	(2.)	(3.)	(4.)

where

(1.): *for $p_k \le 1$ and $s_k \ge 1$ $(k = 0, 1, \dots)$, (1.1) where*
(1.1) $\sup_k \sum_{n=0}^{\infty} |\sum_{j=k}^{\infty} a_{nj} B^{-1/p_k}|^{s_n} < \infty$ *for some $B > 1$;*

(2.): *for $p_k \le 1$ $(k = 0, 1, \dots)$, (2.1) where*
(2.1) $\sup_n (\sup_k |\sum_{j=k}^{\infty} a_{nj}| B^{-1/p_k})^{s_n} < \infty$ *for some $B > 1$*

for $p_k > 1$ $(k = 0, 1, \dots)$, (2.2) and (2.3) where
(2.2) $\sup_n \sum_{k=0}^{\infty} |\sum_{j=k}^{\infty} a_{nj}|^{q_k} B^{-q_k/s_n} < \infty$ *for some $B > 1$*

(2.3) $\begin{cases} \text{for each } n \text{ there is } N_n > 1 \text{ such that} \\ \sup_m \sum_{k=0}^{m} |\sum_{j=m}^{\infty} a_{nj}|^{q_k} N_n^{-q_k} < \infty; \end{cases}$

(3.): *for $p_k \le 1$ $(k = 0, 1, \dots)$, (3.1) and (3.2) where*
(3.1) $\lim_{n\to\infty} |\sum_{j=k}^{\infty} a_{nj}|^{s_n} = 0$ *for each k*

(3.2) $\lim_{M\to\infty} \limsup_{n\to\infty} (\sup_k |\sum_{j=k}^{\infty} a_{nj}| M^{-1/p_k})^{s_n} = 0$

for $p_k > 1$ $(k = 0, 1, \dots)$, (2.3), (3.1) and (3.3) where
(3.3) $\begin{cases} \lim_{M\to\infty} \limsup_{n\to\infty} (\sum_{k=0}^{\infty} |\sum_{j=k}^{\infty} a_{nj}|^{q_k} B^{q_k/s_n} M^{-q_k})^{s_n} = 0 \\ \text{for all } B \ge 1; \end{cases}$

(4.): *for* $p_k \leq 1$ $(k = 0, 1, \dots)$ *(4.1) and*
there exists a sequence $(\alpha_k)_{k=0}^{\infty}$ *such that (4.2) and (4.3)*
where
(4.1) $\sup_{n,k} |\sum_{j=k}^{\infty} a_{nj}| B^{-1/p_k} < \infty$ *for some* $B > 1$
(4.2) $\lim_{n\to\infty} |\sum_{j=k}^{\infty} a_{nj} - \alpha_k|^{s_n} = 0$ *for each* k
(4.3) $\lim_{M\to\infty} \limsup_{n\to\infty} (\sup_k |\sum_{j=k}^{\infty} a_{nj} - \alpha_k| M^{-1/p_k})^{s_n} = 0;$
for $p_k > 1$ $(k = 0, 1, \dots)$, *(2.3), (4.4) and*
there exists a sequence $(\alpha_k)_{k=0}^{\infty}$ *such that (4.2) and (4.5)*
where
(4.4) $\sup_n \sum_{k=0}^{\infty} |\sum_{j=k}^{\infty} a_{nj}|^{q_k} B^{-q_k} < \infty$ *for some* $B > 1$
(4.5) $\begin{cases} \lim_{M\to\infty} \limsup_{n\to\infty} (\sum_{k=0}^{\infty} \sum_{j=k}^{\infty} |a_{nj} - \alpha_k|^{q_k} B^{q_k/s_n} M^{-q_k})^{s_n} \\ = 0 \qquad \text{for all } B \geq 1 \end{cases}.$

Proof. In each case, we apply [12, Theorem 2.7] and Theorem 2.2 for $p_k \leq 1$ or Theorem 2.1 for $p_k > 1$. Then **(1.)** follows from [2, Theorem 5.1.1], **(2.)** from [8, Theorem 5(i)] for $p_k \leq 1$ and [8, Theorem 7] for $p_k > 1$, **(3.)** from [8, Theorem 5 (ii)] for $p_k < 1$ and [8, Theorem 8] for $p_k > 1$, and **(4.)** follows from [8, Theorem 5 (iii)] for $p_k \leq 1$ and [8, Theorem 9] for $p_k > 1$. □

Theorem 3.2

Let $p \in \ell_\infty$. *Then the conditions for* $A \in (bv(p), Y)$ *for* $Y = \ell_1, \ell_\infty, c_0, c$ *can be read from the table*

To / From	ℓ_1	ℓ_∞	c_0	c
$bv(p)$	(1.)	(2.)	(3.)	(4.)

where

(1.): *for* $p_k \leq 1$ $(k = 0, 1, \dots)$, *(1.1) where*
(1.1) $\sup_k \sum_{n=0}^{\infty} |\sum_{j=k}^{\infty} a_{nj}| B^{-1/p_k} < \infty$ *for some* $B > 1$
for $p_k > 1$ $(k = 0, 1, \dots)$, *(1.2) and (1.3) where*
(1.2) $\sup_{\substack{N \subset \mathbb{N}_0 \\ N \text{finite}}} \sum_{k=0}^{\infty} |\sum_{n\in N} \sum_{j=k}^{\infty} a_{nj}|^{q_k} B^{-q_k} < \infty$ *for some* $B > 1$
(1.3) is (2.3) in Theorem 3.1;

(2.): *for* $p_k \leq 1$ $(k = 0, 1, \dots)$, *(2.1) where*
(2.1) $\sup_{n,k} |\sum_{j=k}^{\infty} a_{nj}|^{p_k} < \infty$
for $p_k > 1$ $(k = 0, 1, \dots)$, *(1.3) and (2.2) where*
(2.2) $\sup_n \sum_{k=0}^{\infty} |\sum_{j=k}^{\infty} a_{nj}|^{q_k} B^{-q_k} < \infty$ *or some* $B > 1;$

(3.): *for* $p_k \leq 1$ $(k = 0, 1, \dots)$, *(2.1) and (3.1) where*
(3.1) $\lim_{n\to\infty} \sum_{j=k}^{\infty} a_{nj} = 0$ *for each* k
for $p_k > 1$ $(k = 0, 1, \dots)$, *(1.3), (2.2) and (3.1)*

(4.): *for $p_k \le 1$ $(k = 0, 1, \dots)$, (2.1) and (4.1) where*
(4.1) $\lim_{n\to\infty} \sum_{j=k}^{\infty} a_{nj} = \alpha_k$ *for each k*
for $p_k > 1$ $(k = 0, 1, \dots)$, (1.3), (2.2) and (4.1).

Proof. For $p_k \le 1$, **(1.)** is an immediate consequence of Theorem 3.1 **(1.)**. In the other cases, we apply [12, Theorem 2.7] and Theorem 2.2 for $p_k \le 1$ or Theorem 2.1. Then the case $p_k > 1$ follows from [9, Satz 1]. Furthermore **(2.)**, **(3.)** and **(4.)** follow from [3, Theorem 1], [3, Theorem 1] and [17, 8.3.6, p. 123], and from [3, Corollary], respectively. □

Now we apply Theorems 3.1 and 3.2 to obtain the mapping theorems in [16] and [13].

Remark 6. Putting $p = e$ in Theorem 3.1 **(1.)**, **(2.)**, **(3.)** and **(4.)**, we obtain [16, Theorems 1, 2, 3, and 4], respectively. Furthermore, [16, Corollaries 2^6 (ii), p. 58, 1^6 (i), p. 59 and Corollaries 2^6, p. 60 and p. 61] follow from Theorem 3.2 **(1.)**, **(2.)** **(3.)** and **(4.)**, respectively. In fact, condition (3.1) is replaced in [16, Corollary 2^6, p. 60] by the conditions $\lim_{n\to\infty} a_{nk} = 0$ for each k and $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0$, which are obviously equivalent with condition (3.1), since $a_{nk} = \sum_{j=k}^{\infty} a_{nj} - \sum_{j=k+1}^{\infty} a_{nj}$ for all k, and also $\sum_{j=k}^{\infty} a_{nj} = \sum_{j=0}^{\infty} a_{nj} - \sum_{j=0}^{k-1} a_{nj}$ for all k; a similar remark applies to the conditions in [16, Corollary 2^6, p. 61]. Finally, [13, Theorems 3.2 and 3.1 (a), (b) and (c)] are an immediate consequence of Theorem 3.2 **(1.)**, **(2.)**, **(3.)** and **(4.)**, respectively.

Remark 7. In view of [11, Theorem 1], we obtain the characterizations of the classes (X, Y) for $X = bv(p)$ and $Y = bv(s)$, $Y = Z(\Delta) = \{y \in \omega : \Delta y \in Z\}$ for $Z = \ell_\infty(s), c_0(s), c(s), \ell_1, \ell_\infty, c_0$ and c, if we replace the terms a_{nj} by $c_{nj} = a_{nj} - a_{n-1,j}$ $(n, j = 0, 1, \dots)$ in the respective conditions of Theorems 3.1 and 3.2.

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