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Geometry of arithmetically Gorenstein curves in $\mathbb{P}^4$

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Abstract

We characterize the postulation character of arithmetically Gorenstein curves in $\mathbb{P}^4$. We give conditions under which the curve can be realized in the form $mH - K$ on some ACM surface. Finally, we complement a theorem of Watanabe by showing that any general arithmetically Gorenstein curve in $\mathbb{P}^4$ with arbitrary fixed postulation character can be obtained from a line by a series of ascending complete-intersection biliaisons.

Introduction

In this paper we illustrate some general results about arithmetically Gorenstein (AG) schemes in codimension 3 by a closer analysis of the geometry of AG curves in $\mathbb{P}^4$. We give a numerical criterion for when an AG curve Y in $\mathbb{P}^4$ can be obtained in the form $mH - K$ on an ACM surface X whose postulation character is the expected "first half" of the postulation character of Y. We also give examples of AG curves that cannot be obtained in this form on any ACM surface. Then we prove a theorem showing that a general AG curve with a given postulation character γ can be obtained by ascending CI-biliaison from a line, complementing the result of Watanabe that any codimension 3 AG scheme is licci.

Section 1 contains a review of the structure of codimension 2 ACM schemes in $\mathbb{P}^N$. Section 2 contains a review of known results about codimension 3 AG schemes in $\mathbb{P}^N$. Section 3 contains the study of AG curves of the form $mH - K$ on an ACM surface, together with examples. Section 4 contains the theorem about ascending CI-biliaisons.

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1. ACM codimension 2 subschemes of $\mathbb{P}^N$

There are in the literature many ways of recording the numerical information associated with a subscheme of projective space. Ellingsrud [4] uses a *numerical type* (n_{ij}) associated with a resolution of the ideal. Gruson and Peskine [5] use a *numerical character* associated with the projection on a hyperplane. Then there is the *Hilbert function* $H(n) = h^0(\mathcal{O}_X(n))$ of $X \subseteq \mathbb{P}^N$. If $r = \dim X$, the difference function $\partial^{r+1}H(n)$ is called the *h-vector* of X [15, §1.4]. Finally, there is the *postulation character*, also called *γ-character* of [14]. The numerical information in each of these is more or less equivalent, but unfortunately the terminology varies from one place to another. We find it most convenient, following [14] and [18] to use the postulation character.

DEFINITION. Let X be a closed subscheme of $\mathbb{P}^N_k$ with ideal sheaf $\mathcal{I}_X$. Let $\varphi(n) = h^0(\mathcal{O}_{\mathbb{P}^N}(n)) - h^0(\mathcal{I}_X(n))$ be the postulation function of X. We then define the *postulation character* or *γ-character* of X to be $\gamma_X(n) = -\partial^N\varphi(n)$, the N-th difference function.

Proposition 1.1 [14], [18]

For any proper closed subscheme $X \subseteq \mathbb{P}^N$, the γ-character has the following properties

a) $\gamma(n) = 0$ *for* $n < 0$.
b) $\gamma(n) = -1$ *for* $0 \le n < s$, *where* $s = \min\{n \mid h^0(\mathcal{I}_X(n)) \neq 0\}$, *namely, the least degree of a hypersurface containing* X.
c) $\gamma(s) \geq 0$.
d) $\sum_{n\in\mathbb{Z}} \gamma(n) = 0$.

Note that the function $\varphi(n)$, and hence the Hilbert polynomial of X, can be recovered by numerical integration so that, for example, the degree and genus of a curve can be expressed in terms of its γ-character.

For an ACM subscheme of $\mathbb{P}^N$ of dimension ≥ 1 we have $H^1(\mathcal{I}_X(n)) = 0$ for all n, so to know the γ-character is equivalent to knowing the Hilbert function $h^0(\mathcal{O}_X(n))$.

For ACM subschemes in codimension 2 one has precise information about the possible γ-characters. We call a numerical function $\gamma(n)$ *admissible* if it satisfies the four conditions of (1.1) for some positive integer s.

Theorem 1.2

a) *Let X be a codimension 2 ACM subscheme of $\mathbb{P}^N$. Then its γ-character is positive in the sense that $\gamma(n) \geq 0$ for all $n \geq s$.*
b) *Conversely, given an admissible numerical function $\gamma(n)$ that is positive, as defined in* a)*, there exists a codimension 2 ACM subscheme X in $\mathbb{P}^N$ with that γ-character.*
c) *If, furthermore, X is integral, then γ_X is connected in the sense that $\{n \mid \gamma(n) > 0\}$ is a connected set of integers.*
d) *If γ is a positive connected numerical function, then there exists an integral ACM codimension 2 subscheme $X \subseteq \mathbb{P}^N$ with that γ-character for all $N \geq 3$. If $N = 3$ or 4, X can be taken to be a smooth curve or surface, respectively.*

Proof. These results are due to Gruson and Peskine [5]; see also [14, pp. 34,111], [15], and [18]. □

Remark. The condition "γ connected" is equivalent to the condition that the numerical character of [5] should have *no gaps*, and the condition that the h-vector should be *of decreasing type* [15, pp. 32,97]. It is also equivalent to the condition $m(X) \geq 3$ of Sauer (see [18, p. 452]).

We have also the results of Ellingsrud about the Hilbert scheme and the theorem of Gaeta.

Theorem 1.3 [4]

For any positive admissible γ-character, the set of all ACM codimension 2 subschemes of $\mathbb{P}^N$ is a smooth, irreducible, open subset of the Hilbert scheme of all closed subschemes of $\mathbb{P}^N$. (There is also an explicit formula for its dimension.)

Theorem 1.4 (Gaeta: see [19])

A codimension 2 subscheme $X \subseteq \mathbb{P}^N$ is ACM if and only if it is in the liaison equivalence class of a complete intersection.

2. AG codimension 3 subschemes of $\mathbb{P}^N$

Here we review the analogous results for arithmetically Gorenstein subschemes of codimension 3 of $\mathbb{P}^N$. See also [15, §4.3] for a summary of these results.

A closed subscheme $X \subseteq \mathbb{P}^N_k$ is *arithmetically Gorenstein* (AG) if its homogeneous coordinate ring is a Gorenstein ring. If $\dim X \geq 1$, this is equivalent to saying X is ACM and its canonical sheaf ω_X is isomorphic to $\mathcal{O}_X(m)$ for some $m \in \mathbb{Z}$.

Watanabe [21] showed that the homogeneous ideal of a codimension 3 AG scheme is minimally generated by an odd number of elements. His method of proof also allows one to deduce the following result, neither stated nor proved in his paper, but usually attributed to him.

Proposition 2.1 (Watanabe)

Any AG codimension 3 subscheme of $\mathbb{P}^N$ is in the CI-liaison class of a complete intersection.

Buchsbaum and Eisenbud [1] explained the theorems of Watanabe by giving a structure theorem for Gorenstein codimension 3 algebras from which all further results about AG codimension 3 schemes are deduced.

Theorem 2.2 [1]

The homogeneous ideal of any AG codimension 3 subscheme of $\mathbb{P}^N$ is generated by the Pfaffians of the $(n-1) \times (n-1)$ minors of a certain skew symmetric matrix of homogeneous polynomials, of odd rank n.

Stanley [20], drawing on old results of Macaulay, and applying the theorem of Buchsbaum and Eisenbud, characterized the possible h-vectors of AG codimension 3 subschemes. Translated into the language of the γ-character, his result is this.

Proposition 2.3 [20]

An admissible numerical function γ is the γ-character of an AG codimension 3 subscheme of $\mathbb{P}^N$ if and only if

a) *it is symmetric, meaning there exists an integer q such that $\gamma(n) = \gamma(q-n)$ for all $n \in \mathbb{Z}$ (which implies that $q = \max\{n \mid \gamma(n) \neq 0\}$), and*
b) *if we define the δ-character to be the "first half" of γ, namely*

$$\delta(n) = \begin{cases} \gamma(n) & \text{for } n < q/2 \\ \frac{1}{2}\gamma(n) & \text{for } n = q/2 \text{ if } q \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

then δ is a positive admissible function, as in (1.1).

Proof. For a codimension 3 subscheme of $\mathbb{P}^N$, the γ-character is the negative of the second difference function of the h-vector. So the symmetry of the h-vector in Stanley's theorem [20, 4.2] is equivalent to the symmetry of γ. For the second condition, Stanley says the first half of the first difference function of the h-vector should be an O-sequence. This says it is the h-vector of a codimension 2 zero-dimensional scheme [20, 2.2]. But we know these by (1.2), namely δ should be admissible and positive. We must take $\delta(n) = \frac{1}{2}\gamma(n)$ for $n = q/2$ to make δ be the negative second difference function of the first half of the h-vector corresponding to γ. □

Proposition 2.4 [16]

AG codimension 3 subschemes of $\mathbb{P}^N$, for $N \geq 4$, are parametrized by a smooth, open subset of the Hilbert scheme.

Proposition 2.5 [3]

The AG codimension 3 subschemes of $\mathbb{P}^N$ with a fixed Hilbert function (i.e., with a fixed γ-character as in (2.3)*) form an irreducible subset of the Hilbert scheme. (And Kleppe and Miró–Roig [13] have given a formula for the dimension of this Hilbert scheme.)*

For the existence of integral AG codimension 3 schemes in $\mathbb{P}^N$, Herzog, Trung, and Valla gave a condition in terms of the degree matrix of the defining skew symmetric matrix of (2.2). Then De Negri and Valla translated this condition in terms of the h-vector. Combining their results and stating them with the γ-character, we have the following.

Theorem 2.6 [11], [2]

a) *If X is an integral AG codimension 3 subscheme of $\mathbb{P}^N$, then its δ-character is connected* (1.2*c*).

b) *Conversely, if γ is a numerical function satisfying the conditions of* (2.3) *with δ connected, then there exists a normal integral AG codimension 3 subscheme of $\mathbb{P}^N$ with that γ-character (for $N \geq 4$). In particular, if $N = 4$, we may take X to be a smooth curve.*

3. Arithmetically Gorenstein curves in $\mathbb{P}^4$

Now we will look in more detail at the situation of curves in $\mathbb{P}^4$. If Y is an AG curve in $\mathbb{P}^4$ with postulation character γ_Y, we know from (2.3) that its "first half" δ_Y is the postulation character of an ACM surface X in $\mathbb{P}^4$. So our first task will be to explore the relationship between the ACM surfaces X with postulation character δ_Y and the AG curve Y. There is a well-known method of obtaining an AG curve on an ACM surface.

Proposition 3.1

Let X be an ACM surface in $\mathbb{P}^4$ satisfying the additional condition G_1, Gorenstein in codimension 1. Let K be the canonical divisor, and let Y be an effective divisor linearly equivalent to $mH - K$ for some $m \in \mathbb{Z}$. Then Y is an AG curve with $\omega_Y \cong \mathcal{O}_Y(m)$. If X satisfies only G_0, then the canonical divisor K may not be defined, but there is an anticanonical divisor M, and the same is true for an effective divisor $Y \sim mH + M$.

Proof. This construction was given in [12, 5.4]. See also [15, 4.2.8]. The extension to the case where X satisfies only G_0 is in [10]. □

Lemma 3.2

Let X be an ACM surface in $\mathbb{P}^4$ with postulation character γ_X, and let $r = \max\{n \mid \gamma_X(n) \neq 0\}$. Then

1) $H^2(\mathcal{O}_X(n)) = 0$ *for all* $n \geq r - 3$.
2) $\mathcal{I}_X$ *is r-regular.*
3) $\mathcal{I}_X(r)$ *is generated by global sections.*
4) $\mathcal{I}_X$ *has a resolution*

$$0 \to \oplus \mathcal{O}_{\mathbb{P}^4}(-b_j) \to \oplus \mathcal{O}_{\mathbb{P}^4}(-a_i) \to \mathcal{I}_X \to 0$$

with $\max\{b_j\} = r + 1$.

Proof. Since X is ACM, we have $H^1(\mathcal{O}_X(n)) = 0$ for all n. Hence the Euler characteristic $\chi(\mathcal{O}_X(n)) = h^0(\mathcal{O}_X(n)) + h^2(\mathcal{O}_X(n))$, and this is equal to the Hilbert polynomial of X. When we take difference functions of $h^0(\mathcal{O}_X(n))$, the third and fourth differences will be 0 if and only if the corresponding shift of $h^0(\mathcal{O}_X(n))$ is equal to the polynomial $\chi(\mathcal{O}_X(n))$. We conclude that $\gamma_X(n) = 0$ for $n \geq r+1$ is equivalent to $h^2(\mathcal{O}_X(n)) = 0$ for $n \geq r-3$.

Since $\mathcal{I}_X$ has no H^1 or H^2, because of X being ACM, and $H^3(\mathcal{I}_X(n)) \cong H^2(\mathcal{O}_X(n))$, we find $\mathcal{I}_X$ is r-regular. This implies $\mathcal{I}(r)$ generated by global sections, by the theorem of Castelnuovo–Mumford.

Finally, take a minimal resolution of I_X over the homogeneous coordinate ring, and sheafify. This gives an exact sequence of cohomology

$$0 \to H^3(\mathcal{I}_X(n)) \to \oplus H^4(\mathcal{O}_{\mathbb{P}^4}(n-b_j)) \xrightarrow{\alpha} \oplus H^4(\mathcal{O}_{\mathbb{P}^4}(n-a_i)).$$

Because of the minimality of the resolution, $\max\{b_j\} > \max\{a_i\}$. Hence the largest n for which $H^3(\mathcal{I}_X(n)) \neq 0$ is equal to the largest n for which some $H^4(\mathcal{O}_{\mathbb{P}^4}(n-b_j)) \neq 0$. We conclude $n \geq r-3$ if and only if $n - b_j > 5$ for all j, and hence $\max\{b_j\} = r+1$. □

Lemma 3.3

Let X be a locally complete intersection surface in $\mathbb{P}^4$, with ideal sheaf $\mathcal{I}$. Then $\Lambda^2(\mathcal{I}/\mathcal{I}^2) \cong \omega_X^{\vee}(-5)$.

Proof. [6, III.7.11]. □

Lemma 3.4

Let Y be an AG curve in $\mathbb{P}^4$ with $\omega_Y \cong \mathcal{O}_Y(m)$. Then for all $n \in \mathbb{Z}$, $\gamma_Y(n) = \gamma_Y(m+4-n)$. In other words γ_Y is symmetric with the q of (2.3a) equal to $m+4$.

Proof. By duality on Y we have $H^2(\mathcal{O}_Y(n))$ dual to $H^0(\mathcal{O}_Y(m-n))$ for all n. Hence by Riemann–Roch,

$$h^0(\mathcal{O}_Y(n)) = dn + 1 - g + h^0(\mathcal{O}_Y(m-n)).$$

When we take the fourth difference function, this gives

$$\gamma_Y(n) = \gamma_Y(m+4-n).$$

Hence γ_Y is symmetric with $q = m+4$. □

Now we can state our main result about the AG curves of the form $mH - K$ on an ACM surface X.

Theorem 3.5

Let X be an ACM surface in $\mathbb{P}^4$ that is locally a complete intersection, and let $r = \max\{n \mid \gamma_X(n) \neq 0\}$.

a) If $m \geq 2r-5$, the linear system $|mH - K|$ is effective and without base points.
b) If $m \geq 2r-4$, then $mH - K$ is very ample, and for any $Y \in |mH - K|$, with postulation character γ_Y, its first half δ_Y is equal to γ_X.
c) If $m \geq 2r-2$, the set of curves $Y \in |mH - K|$ as X also varies in its family, forms an open subset of the Hilbert scheme of AG curves with postulation character γ_Y.

Proof. We have assumed that X is a local complete intersection so that ω_X will be an invertible sheaf and K a Cartier divisor.

a) By (3.2), $\mathcal{I}_X(r)$ is generated by global sections. Hence the same is true of $(\mathcal{I}/\mathcal{I}^2)(r)$ and also of $\Lambda^2(\mathcal{I}/\mathcal{I}^2(r))$. Using (3.3) this shows that $\omega_X^\vee(2r-5)$ is generated by global sections, so the corresponding linear system is effective and without base points.

b) It follows [6, II. Ex. 7.5] for $m \geq 2r-4$, that $mH-K$ will be very ample. To show that $\delta_Y = \gamma_X$, we proceed as follows. First suppose $m \geq 2r-3$. I claim that $\gamma_X(n) = \gamma_Y(n)$ for $n \leq r$. There is an exact sequence

$$0 \to H^0(\mathcal{I}_{Y,X}(n)) \to H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_Y(n)) \to H^1(\mathcal{I}_{Y,X}(n)) \to \dots .$$

We will show that the two outside terms are zero, hence the middle ones isomorphic. Since $Y \sim mH-K$, $\mathcal{I}_{Y,X}(n) \cong \omega_X(n-m)$. By duality on X, $h^1(\mathcal{I}_{Y,X}(n)) = h^1(\mathcal{O}_X(m-n)) = 0$, since X is ACM. Also $h^0(\mathcal{I}_{Y,X}(n)) = h^2(\mathcal{O}_X(m-n))$. Now our assumptions $m \geq 2r-3$ and $n \leq r$ imply $m-n \geq r-3$, so the h^2 is 0 by (3.2). Thus $h^0(\mathcal{O}_X(n)) = h^0(\mathcal{O}_Y(n))$ and taking difference function $\gamma_X(n) = \gamma_Y(n)$ for $n \leq r$.

Now $q = m+4$ by (3.4), hence $q \leq 2r+1$, so $r < \frac{q}{2}$. So we see that the entire non-zero portion of γ_X is equal to the portion of δ_Y for $n \leq r$. Since both are positive admissible characters, they are equal.

The same argument works for $m \geq 2r-4$ except for the case $m = 2r-4$ and $n = r$. In this case $r = \frac{q}{2}$, and since the characters γ_X and δ_Y are equal for $n < r$, and 0 for $n > r$, it follows that they are also equal for $n = r$ by (1.1d).

c) We follow the method of [9, 3.3]. The proof given there already shows the desired result for $m \gg 0$. Note that in [9] the surface X is supposed smooth, but the same holds for X a locally complete intersection, in which case K will be a Cartier divisor.

The first step is that, since X is ACM, the dimension of the linear system $|Y|$ on X is equal to $h^0(Y, \mathcal{N}_{Y/X})$. To see this, we note first that since Y is a Cartier divisor on X, there is an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Y) \to \mathcal{N}_{Y/X} \to 0.$$

Now X is ACM, so $h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_X) = 0$. Thus $h^0(\mathcal{O}_X(Y)) = 1 + h^0(\mathcal{N}_{Y/X})$ and $\dim |Y| = h^0(\mathcal{O}(Y)) - 1 = h^0(\mathcal{N}_{Y/X})$.

The second step is by the theorem of Ellingsrud (1.3) to note that the irreducible component of the Hilbert scheme of ACM surfaces containing X is smooth, and therefore by Grothendieck's differential study of the Hilbert scheme has dimension equal to $h^0(\mathcal{N}_{X/\mathbb{P}^4})$. The third step is to notice that for $m \geq 2r-3$, each such curve Y is contained in a unique such surface X. Indeed, we have seen in the proof of b) above that for $m \geq 2r-3$ we have $\gamma_X(n) = \gamma_Y(n)$ for $n \leq r$, and hence $H^0(\mathcal{I}_X(n)) \to H^0(\mathcal{I}_Y(n))$ is an isomorphism for $n \leq r$. Since $\mathcal{I}_X$ is generated in degrees $\leq r$ (3.2), we see that the homogeneous ideal of X is uniquely determined by Y. It follows now that the dimension of the family of curves $Y \sim mH - K$ as X and Y vary is equal to $h^0(Y, \mathcal{N}_{Y/X}) + h^0(X, \mathcal{N}_{X/\mathbb{P}^4})$.

The next step is to show that for $m \geq 2r-2$ we have $h^0(X, \mathcal{N}_{X/\mathbb{P}^4}) = h^0(Y, \mathcal{N}_{X/\mathbb{P}^4} \otimes \mathcal{O}_Y)$. To prove this, we need to show the vanishing of $h^i(X, \mathcal{N}_{X/\mathbb{P}^4}(-Y))$ for $i = 0, 1$. Since $Y \sim mH - K$, these are equal, using duality on X, to $h^i(X, \mathcal{I}_X/\mathcal{I}_X^2(m))$ for $i = 1, 2$.

Take a minimal resolution of $\mathcal{I}_X$, as in (3.2),

$$0 \to \oplus \mathcal{O}_{\mathbb{P}^4}(-b_j) \to \oplus \mathcal{O}_{\mathbb{P}^4}(-a_i) \to \mathcal{I}_X \to 0.$$

Tensoring with $\mathcal{O}_X(m)$ we get an exact sequence

$$\oplus \mathcal{O}_X(m - b_j) \to \oplus \mathcal{O}_X(m - a_i) \to \mathcal{I}_X/\mathcal{I}_X^2(m) \to 0.$$

From (3.2) we know that $a_i, b_j \leq r + 1$ for all i, j. So the hypothesis $m \geq 2r - 2$ implies that $m - a_i, m - b_j \geq r - 3$ for all i, j. So again by (3.2) it follows that $H^2(\mathcal{O}_X(m - a_i)) = H^2(\mathcal{O}_X(m - b_j)) = 0$ for all i, j. Since X is ACM, we have $H^1(\mathcal{O}_X(n)) = 0$ for all n, and now it follows easily that $H^i(\mathcal{I}_X/\mathcal{I}_X^2(m)) = 0$ for $i = 1, 2$.

Now as in the proof of [9, 3.3] we find that

$$h^0(\mathcal{N}_{Y/\mathbb{P}^4}) \leq h^0(\mathcal{N}_{Y/X}) + h^0(\mathcal{N}_{X/\mathbb{P}^4}).$$

The other inequality comes from the fact that the dimension of the family of all AG curves with the same γ-character is less than or equal to $h^0(\mathcal{N}_{Y/\mathbb{P}^4})$, by the differential study of the Hilbert scheme. We conclude equality, so the two families have the same dimension, and the curves of the form $Y \sim mH - K$ with Y, X varying form an open subset of the Hilbert scheme of AG curves containing Y, as required. $\square$

Corollary 3.6

a) *For each numerical function γ satisfying the numerical conditions of* (2.3), *there is an AG curve Y with postulation character γ, lying on an ACM surface X with postulation character $\gamma_X = \delta_Y$.*

b) *If, furthermore, the first half δ of γ is connected, we may take both X and Y to be smooth.*

c) *If the integers m and r associated with γ satisfy $m \geq 2r - 2$, then there is an open subset V of the Hilbert scheme H_γ of AG curves with character γ, such that every $Y \in V$ is of the form $Y \sim mH - K$ on an ACM surface X with character δ, the first half of γ.*

Proof. Note that this corollary gives an independent proof of the existence results for AG curves quoted in (2.3) and (2.6). To prove the corollary, given γ, let δ be its first half. Then there exists a reduced, locally complete intersection ACM surface in $\mathbb{P}^4$ with postulation character δ [18, 3.2]. Let $q = \max\{n \mid \gamma(n) \neq 0\}$ and take $m = q-4$. If $r = \max\{n \mid \delta(n) \neq 0\}$, then $r \leq q/2$ by definition of the "first half" function, so $m \geq 2r-4$. Then by the theorem, $mH - K$ is very ample, and any curve in the linear system will be AG with postulation character having its first half equal to γ_X. By symmetry, γ is uniquely determined by δ and m, which shows that γ_Y is equal to the γ we started with.

If δ is connected, then X can be taken to be smooth [18, 3.3], and since $mH - K$ is very ample, we can take Y to be smooth by the usual Bertini theorem.

The last statement c) is just a reformulation of (3.5c). □

Now we will give some examples to illustrate Theorem 3.5 and show that its results are sharp.

EXAMPLE 3.7: The linear system $mH - K$ may be effective even for $m < 2r-5$. Let X be the Castelnuovo surface. Then $\gamma_X = -1 \ \ -1\ 0\ 2$, $r = 3$, $2r-5 = 1$. Take $m = 0$. Note that $-K = (3; 1^8)$ in the usual notation for divisor classes on X (see, for example, [8, 3.3]). This class is effective and is represented by plane cubic curves $Y \subseteq X$. They all pass through a ninth point $Q \in X$. Thus the linear system $|Y|$ is not without base points.

EXAMPLE 3.8: If $m < 2r-4$, then δ_Y may not be equal to γ_X. One example is the plane cubic curve Y of the previous (3.7). In this case $\gamma_Y = -1\ 1\ 0\ 1\ \ -1$, and $\delta_Y = -1\ 1$ corresponding to a plane H. This is not surprising since the minimal degree surface containing Y is a plane.

For a more interesting example let X be the Del Pezzo surface. Then $\gamma_X = -1\ \ -1\ 1\ 1$, $r = 3$, $2r-4 = 2$. Take $m = 1$. Then $Y \sim H - K = 2H$, and Y is the complete intersection of three quadric hypersurfaces in $\mathbb{P}^4$. It is the canonical curve of degree 8 and genus 5, and $\gamma_Y = -1\ -1\ 2\ 2\ -1\ -1$, so $\delta_Y = -1\ -1\ 2$, the γ-character of a cubic scroll. Our curve Y is on its surface X of minimal degree, yet its δ_Y belongs to a surface of lower degree. There are curves $Y' \sim H - K$ on the cubic scroll X', also canonical curves $(8,5)$, but these form a proper subfamily of all the AG $(8,5)$ curves: they are the canonical embeddings of trigonal curves of genus 5.

In this example we see that while the family of AG curves Y with $\gamma_Y = -1\ -1\ 2\ 2\ \ -1\ \ -1$ is irreducible, equal to the family of canonical curves of genus 5 in $\mathbb{P}^4$, there are two types: the general one being a complete intersection on the Del Pezzo surface and the special one lying on a cubic scroll. In both cases the associated character δ_Y is that of a cubic scroll.

EXAMPLE 3.9: We saw in the proof of (3.5c) that for $m \geq 2r-3$, the curve $Y \sim mH-K$ is contained in a unique ACM surface X with $\gamma_X = \delta_Y$. Here we show that for $m = 2r-4$, the surface X may not be unique.

Recall first that if X, X' are two ACM surfaces without common components, whose union is a complete intersection Z of hypersurfaces of degrees a, b, then the intersection $Y = X \cap X'$ is an AG curve [15, 4.2.1]. In fact, I claim $Y \sim (a+b-5)H+M$ on X where M is the anticanonical divisor. Indeed, since $X \cup X' = Z$ is a complete intersection, and X and X' have no common components, the surface X satisfies G_0, so we can speak of the anticanonical divisor M [10]. From the theory of liaison it follows that $\mathcal{I}_{X',Z} = \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_Z)$. But $\mathcal{I}_{X',Z} = \mathcal{I}_{Y,X}$, and $\mathcal{H}om(\mathcal{O}_X, \mathcal{O}_Z) = \omega_X \otimes \omega_Z^\vee$. Thus on X we have $Y \sim (a+b-5)H + M$, since $\omega_Z = \mathcal{O}_Z(a+b-5)$. So it follows from (3.1) that Y is AG with $m = a+b-5$.

Now for our example, let X again be a Castelnuovo surface. This surface is contained in a unique quadric hypersurface F_2, a cone over the nonsingular quadric surface in $\mathbb{P}^3$. The divisor class group of F_2 is $\mathbb{Z} \oplus \mathbb{Z}$, and X is in the class of bidegree $(2,3)$. Let X' be another Castelnuovo surface of bidegree $(3,2)$. Then $X \cup X' = Z$ is a complete intersection of F_2 with a quintic hypersurface F_5. Therefore $Y = X \cap X'$ is in the class of $2H - K$, so this Y has $m = 2$, and is not contained in a unique Castelnuovo surface X.

We observe also that X passes through the singular point of F_2, since by Klein's theorem it cannot be a Cartier divisor on F_2, and this point is none other than the point Q mentioned above in (3.7). Indeed, the plane cubic curve of (3.7) is contained in a plane Π. This plane intersects F_2 in at least a plane cubic curve, so $\Pi \subseteq F_2$, and Π must also contain the singular point. As the plane cubic curve moves in a pencil, so does Π, and the only point in common is Q, which must therefore be the singular point of F_2.

The same argument shows that X' also contains Q, and so all the curves $Y \sim 2H - K$ obtained by the construction $X \cap X'$ for various X' contain this same point Q. On the other hand, the linear system $|2H - K|$ is very ample by (3.5b), so we see that the curves Y obtained as $X \cap X'$ for linked Castelnuovo surfaces X and X' are not general among all curves in the linear system $|2H - K|$.

EXAMPLE 3.10: We give an example to show that (3.5c) is sharp, namely an example of curves with $m = 2r - 3$ on an ACM surface X that are not general in their Hilbert scheme. Let X be a Castelnuovo surface, with $\gamma_X = -1 \ -1\ 0\ 2$. Then $r = 3$. We take $m = 2r - 3 = 3$, and consider curves $Y \sim 3H - K$ on X. These have character $\gamma = -1 \ -1\ 0\ 2\ 2\ 0\ -1\ -1$ and have degree 18 and genus 28. Each such curve Y is contained in a unique quadric hypersurface F_2, which is the same one that contains the Castelnuovo surface X, and therefore is singular, by Klein's theorem.

On the other hand, there are AG curves Y of degree 18 and genus 28 of the form $Y \sim 3H - K$ on the sextic $K3$ surface X, which is a complete intersection of any quadric and cubic hypersurfaces, $X = F_2 \cap F_3$. In this case we can take F_2 to be smooth, so that the unique quadric hypersurface F_2 containing Y is smooth, and so Y is not on a Castelnuovo surface. Thus the family of $Y \sim 3H - K$ on Castelnuovo surfaces is special in H_γ.

EXAMPLE 3.11: For our last example we show that for certain γ, the general AG curve with postulation character γ is not of the form $mH - K$ on any ACM surface.

Take $\gamma = -1\ -1\ -1\ 6\ -1\ -1\ -1$. Then $\delta = -1\ -1\ -1\ 3$ is the postulation character of a Bordiga surface. There are curves $Y \sim 2H - K = (11; 3^{10})$ on a Bordiga surface X. These curves have degree 14 and genus 15. The dimension of the family of all such Y on Bordiga surfaces X is less than or equal to the dimension of the linear system $|Y|$ on X plus the dimension of the family $\{X\}$ of all Bordiga surfaces. Now $\dim_X |Y| = h^0(\mathcal{N}_{Y/X}) = h^0(\mathcal{O}_Y(Y))$. We calculate $Y^2 = 31$ from its divisor class representation on X. Since $31 > 2g_Y - 2$, the linear system $\mathcal{O}_Y(Y)$ is nonspecial, and $h^0(\mathcal{O}_Y(Y)) = 31 + 1 - 15 = 17$. The dimension of the family $\{X\}$ is 36, by [4]. Thus the family of all $Y \sim 2H - K$ on Bordiga surfaces has dimension $\leq 17 + 36 = 53$.

On the other hand, from the general theory of the Hilbert scheme, we know that the dimension of any irreducible component of the Hilbert scheme of curves of degree d and genus g is $\geq 5d + 1 - g$. In our case, this gives 56. (In fact, the dimension is exactly 56 [13].) Thus the general AG curve Y with given γ cannot be of the form $2H - K$ on a Bordiga surface.

It remains to show that Y cannot be of the form $mH - K$ on any other ACM surface. If X is an ACM surface of degree δ and sectional genus π, and if $Y \sim mH - K$, then we can easily compute the degree d of Y as $(Y.H)$ on the surface, and this gives $d = (m+1)\delta - 2\pi + 2$. In our case, since $\gamma = -1 \ \ -1 \ \ -1 \ 6 \ \ -1 \ \ -1 \ \ -1$, we have $q = 6$ and $m = 2$. So we must have $d = 3\delta - 2\pi + 2$. Now looking at the possible pairs (δ, π), which will be the degree and genus of a nondegenerate ACM curve in $\mathbb{P}^3$, we see (left to reader) that $3\delta - 2\pi + 2 \leq 14$ always with equality only for $(\delta, \pi) = (6, 3)$. So the only way to obtain the curve Y as $mH - K$ on an ACM surface is as $2H - K$ on a Bordiga surface.

R. Miró–Roig [17] has verified by a dimension count that there are similar examples of AG curves of arbitrarily high degree that cannot be obtained in the form $mH - K$ on any ACM surface.

Problem 3.12. For those postulation characters γ of AG curves for which the associated m and r satisfy $m = 2r - 3$ or $m = 2r - 4$, give a stratification of the Hilbert scheme of AG curves with character γ, according to the least degree of an ACM surface containing the curve, the gonality of the abstract curve, which ones are of the form $mH - K$ on an ACM surface, and the dimensions of the strata, so as to generalize and complete the information illustrated in examples (3.8), (3.10), and (3.11) above.

4. Complete intersection biliaison

If C is a curve in $\mathbb{P}^4$, recall that a *complete intersection* (CI) *biliaison* of C is obtained by taking a complete intersection surface $X \subseteq \mathbb{P}^4$, and taking a curve $C' \sim C + hH$ on X, where H is the hyperplane class, and $h \in \mathbb{Z}$. It is *ascending* if $h \geq 0$. The equivalence relation generated by these is called CI-*biliaison* and it is equivalent to even CI-liaison [7, 4.4].

In this section we will show that a general AG curve in $\mathbb{P}^4$ with an arbitrary postulation character is obtained by ascending CI-biliaisons from a line. This provides a new proof and strengthening of Watanabe's result (2.1) for general AG curves.

Lemma 4.1

Let Y be an AG curve in $\mathbb{P}^4$ with postulation character γ. Let $s = \min\{n > 0 \mid \gamma(n) \geq 0\}$, let $q = \max\{n \mid \gamma(n) \neq 0\}$, and let m be the integer for which $\omega_Y \cong \mathcal{O}_Y(m)$. Then

a) $m = q - 4$.
b) $\mathcal{I}_Y(q-s)$ *is generated by global sections.*
c) $\mathcal{I}_Y$ *is* $(q-1)$*-regular.*

Proof. Part a) we recall for memory (3.4). For parts b), c) we use the theorem of Buchsbaum–Eisenbud [1] in the notation of [11, §5, pp. 62-63]. Let S be the homogeneous coordinate ring of $\mathbb{P}^4$. Then the homogeneous ideal I_Y of Y has a resolution of the form

$$0 \to S(-c) \to \oplus S(-b_i) \to \oplus S(-a_i) \to I_Y \to 0$$

with $i = 1, 2, \ldots, 2r+1$ for some positive integer r. Moreover, this resolution is *symmetric* in the sense that if we order $a_1 \leq a_2 \leq \ldots \leq a_{2r+1}$ and $b_1 \geq b_2 \geq \ldots \geq b_{2r+1}$, then $b_i = c - a_i$ for each i. Furthermore, if we let $u_{ij} = b_i - a_j$ be the associated degree matrix, then $u_{ij} > 0$ for $i + j = 2r + 3$.

To relate this to the invariants s and q of the γ-character, first note that the a_i are the degrees of a minimum set of generators of I_Y. Hence $a_1 = s$, which is the least degree of a generator. By symmetry, $b_1 = c - s$. Computing $\omega_Y \cong \mathcal{E}xt^3(\mathcal{O}_Y, \omega_{\mathbb{P}^4})$ using this resolution, we find $\omega_Y \cong \mathcal{O}_Y(c-5)$. Hence $m = c - 5$ and $q = c - 1$. From the inequality $u_{2,2r+1} > 0$ we find $b_2 > a_{2r+1} = \max\{a_i\}$. But $b_1 = c - s \geq b_2$, so we find $\max\{a_i\} < c - s$. Hence $\max\{a_i\} \leq q - s$, and $\mathcal{I}_Y(q-s)$ is generated by global sections.

Finally, to show that $\mathcal{I}_Y$ is $(q-1)$-regular, we use this resolution to show $h^2(\mathcal{I}_Y(q-3)) = 0$ by climbing up the resolution and using the fact that $h^4(\mathcal{O}_{\mathbb{P}}(-c+q-3)) = h^4(\mathcal{O}_{\mathbb{P}}(-4)) = 0$. $\square$

Theorem 4.2

For any postulation character γ corresponding to an AG curve in $\mathbb{P}^4$ (as in (2.3)), there is a nonempty open subset V_γ of the corresponding Hilbert scheme H_γ of these curves, such that any curve $Y \in V_\gamma$ can be obtained by strictly ascending CI-biliaisons from a line in $\mathbb{P}^4$.

Proof. We will prove by induction on the degree, the following slightly more precise statement. For each γ, there is a nonempty open set $V_\gamma \subseteq H_\gamma$ such that for any $Y \in V_\gamma$

(i) There is a complete intersection surface $X = F_s \cap F_{q-s}$ containing Y that is reduced, and such that for each irreducible component U_i of $X \backslash \mathrm{Sing}\, X$, the intersection $Y \cap U_i \neq \emptyset$, and
(ii) There is an AG curve $Y' \sim Y - H$ on X, with postulation character γ', such that $Y' \in V_{\gamma'}$.

To begin with, by definition of s, Y is contained in a hypersurface F_s of degree s. Since $\mathcal{I}_Y(q-s)$ is generated by global sections, there is a hypersurface F_{q-s} containing

Y, whose intersection with F_s is a surface X. Thus every $Y \in H_\gamma$ is contained in a complete intersection surface $X_{s(q-s)}$, and so the property (i) is an open (possibly empty) condition on H_γ. The fact that V_γ is nonempty will appear in the induction step below.

We start the induction with AG curves Y having $s = 1$. These are contained in a $\mathbb{P}^3$, so they are complete intersection curves, and for these the theorem is immediate. We descend by the biliaison of (ii) unless Y is a plane curve, in which case we do biliaisons in the plane containing Y.

So now we assume $s \geq 2$. Suppose for a moment that $Y \subseteq X$ satisfies condition (i). We will show that the linear system $|Y - H|$ is nonempty and contains a unique AG curve Y'. We use the exact sequence of [7, 2.10], twisted by $-H$:

$$0 \to \mathcal{O}_X(-H) \to \mathcal{L}(Y - H) \to \omega_Y \otimes \omega_X^\vee(-H) \to 0.$$

Now $\omega_X \cong \mathcal{O}_X(s + q - s - 5) = \mathcal{O}_X(q - 5)$, and $\omega_Y = \mathcal{O}_Y(m) = \mathcal{O}_Y(q - 4)$, so the sheaf on the right is just $\mathcal{O}_Y$. Since $h^0(\mathcal{O}_X(-H)) = h^1(\mathcal{O}_X(-H)) = 0$, we find $h^0(\mathcal{L}(Y - H)) = h^0(\mathcal{O}_Y) = 1$, so it has a unique section t whose restriction to Y is 1. From the condition that Y meets every irreducible component of $X \backslash \mathrm{Sing}\, X$, and X being reduced, we conclude that t is nondegenerate, and defines an effective divisor $Y' \sim Y - H$ [7, 2.9]. The support of the divisor Y' consists of the set of points of X where t does not generate the stalk of the sheaf $\mathcal{L}(Y - H)$. Since t restricted to Y is 1, we find that $Y \cap Y' = \emptyset$ as subsets of X.

Now consider the sequence of [7, 2.10] for Y':

$$0 \to \mathcal{O}_X \to \mathcal{L}(Y') \to \omega_{Y'} \otimes \omega_X^\vee \to 0.$$

Since $Y' \sim Y - H$ and $Y \cap Y' = \emptyset$, the sheaf $\mathcal{L}(Y')$ is invertible isomorphic to $\mathcal{L}(-H)$ on $X \backslash Y$, which is a neighborhood of Y'. Thus $\omega_{Y'} \otimes \omega_X^\vee \cong \mathcal{O}_{Y'}(-H)$. From this we find $\omega_{Y'} \cong \mathcal{O}_{Y'}(q - 6) = \mathcal{O}_{Y'}(m - 2)$. Hence Y', which is ACM by virtue of the biliaison from Y to Y', is arithmetically Gorenstein. We can compute its γ-character

$$\gamma'(n) = \begin{cases} -1 & \text{for } 0 \leq n \leq s - 2 \\ \gamma(s) - 1 & \text{for } n = s - 1, q - s - 1 \\ \gamma(n + 1) & \text{for } s \leq n \leq q - s - 2 \\ -1 & \text{for } q - s \leq n \leq q - 2. \end{cases}$$

Now we explain the induction step of the proof. We will construct, and fix, the open sets $V_\gamma \subseteq H_\gamma$ inductively during the course of the proof. For $s = 1$ we take $V_\gamma = H_\gamma$. Given γ with $s \geq 2$, define a character γ' by the recipe just given. By the induction hypothesis we have already constructed an open set $V_{\gamma'} \subseteq H_{\gamma'}$ of curves satisfying (i) and (ii). Let Y' be such a curve, and let $Y' \subseteq X' = F_{s'} \cap F_{q'-s'}$ satisfy (i). Note that $q' = q - 2$ and s' is either s or $s - 1$. So define a surface $X = (F_{s'} + H_1) \cap (F_{q'-s'} + H_2)$ or $X = F_{s'} \cap (F_{q'-s'} + H_1 + H_2)$, where H_1, H_2 are hyperplanes in general position. Then X is a reduced complete intersection surface of degree $s(q-s)$.

On this surface X, we will show, by an argument analogous to the one above, that a general curve Y in the linear system $Y'+H$ on X is an AG curve. First we write the sequence of [7, 2.10] for Y', twisted by H:

$$0 \to \mathcal{O}_X(H) \to \mathcal{L}(Y'+H) \to \omega_{Y'} \otimes \omega_X^{\vee}(H) \to 0.$$

Knowing that $\omega_{Y'} \cong \mathcal{O}_{Y'}(m-2)$ we see as above that the right-hand sheaf is $\mathcal{O}_{Y'}$. Since X is a complete intersection, $H^1(\mathcal{O}_X(H)) = 0$, so the map $H^0(\mathcal{L}(Y'+H)) \to H^0(\mathcal{O}_{Y'}) \to 0$ is surjective. Since the sheaf $\mathcal{L}(Y'+H)$ has nondegenerate sections (for example corresponding to the trivial biliaison $Y'+H$), there exists a nondegenerate section $s \in H^0(\mathcal{L}(Y'+H))$ whose image in $H^0(\mathcal{O}_{Y'})$ is 1. Let Y be the associated divisor. Then $Y \cap Y' = \emptyset$, and the sequence of [7, 2.10] for Y shows as above, that $\omega_Y \cong \mathcal{O}_Y(m)$. Hence Y is an AG curve. Since the trivial biliaison $Y'+H$ satisfies (i), and this is an open condition, we can choose Y also so that it satisfies (i).

Thus there exists a nonempty open subset of curves $Y \in H_\gamma$ satisfying (i). Since the procedures of constructing Y' from Y and Y from Y' are reversible, we can define the open subset $V_\gamma \subseteq H_\gamma$ to be the set of AG curves Y satisfying (i) with the associated curve Y' lying in $V_{\gamma'}$.

This completes the inductive proof of (i) and (ii). To prove the theorem, we take a $Y \in V_\gamma$, and by (ii) find a $Y' \in V_{\gamma'}$ with smaller degree. We continue this process until either the degree is 1 or $s=1$, which we have discussed above. □

Remark 4.3. If we restrict our attention to nonsingular AG curves Y, the curve will lie on a nonsingular complete intersection surface $X_{s(q-s)}$, and if Y is sufficiently general, the associated curve $Y' \sim Y-H$ will also be nonsingular. Note that if Y has connected δ-character (2.6) then Y' also has connected δ-character. Thus for sufficiently general nonsingular curves Y we can carry out the CI-biliaisons using only nonsingular AG curves lying on nonsingular complete intersection surfaces.

EXAMPLE 4.4: We have seen (3.11) that the general AG curve Y with $\gamma = -1\ \ -1\ \ -1\ 6\ \ -1\ \ -1\ \ -1$, a curve of degree 14 and genus 15, cannot be obtained in the form $mH-K$ on any ACM surface. But applying our theorem, we see that it can be obtained in the form $Y'+H$ on a complete intersection surface $X_{3.3}$, where Y' has $\gamma' = -1\ \ -1\ 4\ \ -1\ \ -1$. This is an AG curve of degree 5 and genus 1, which we may take to be nonsingular when Y is sufficiently general.

The curve Y' in turn lies on a complete intersection $X'_{2.2}$, a Del Pezzo surface, and $Y'' = Y'-H'$ on X' is a line. Thus Y is obtained by two ascending CI-biliaisons from a line.

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