Collectanea Mathematica (electronic version): http://www.imub.ub.es/collect

Collect. Math. **55**, 1 (2004), 37–60

Equivalence of families of singular schemes on threefolds and on ruled fourfolds

Flaminio Flamini

Dipartimento di Matematica, Università degli Studi di L'Aquila

Via Vetoio - Loc. Coppito, 67010 L'Aquila - Italy

E-mail: flamini@mat.uniroma3.it

Received March 4, 2003. Revised June 30, 2003

Abstract

The main purpose of this paper is twofold. We first analyze in detail the meaningful geometric aspect of the method introduced in [12], concerning families of irreducible, nodal "curves" on a smooth, projective threefold X. This analysis gives some geometric interpretations not investigated in [12] and highlights several interesting connections with families of other singular geometric "objects" related to X and to other varieties.

Then, we use this method to study analogous problems for families of singular divisors on ruled fourfolds suitably related to X. This enables us to show that Severi varieties of vector bundles on X can be rephrased in terms of "classical" Severi varieties of divisors on such fourfolds.

Introduction

The theory of families of singular curves with fixed invariants (e.g. geometric genus, singularity type, number of irreducible components, etc.) and contained in a projective variety X has been extensively studied from the beginning of Algebraic Geometry and it actually receives a lot of attention, partially due to its connections with several fields in Geometry and Physics.

Nodal curves play a central role in the subject of singular curves. Families of irreducible and δ-nodal curves on a given projective variety X are usually called *Severi varieties* of irreducible, δ-nodal curves in X. The terminology "Severi variety" is due

Keywords: Nodal curves, threefolds, vector bundles.

MSC2000: 14H10, 14J30; (Secondary): 14J60, 14J35, 14C20.

The author is a member of Cofin GVA, EAGER and GNSAGA-INdAM.

to the classical case of families of nodal curves on $X = \mathbb{P}^2$, which was first studied by Severi (see [23]).

The case in which X is a smooth projective surface has recently given rise to a huge amount of literature (see, for example, [4], [5], [6], [7], [11], [14], [15], [21], [22] just to mention a few. For a chronological overview, the reader is referred for example to Section 2.3 in [10] and to its bibliography). This depends not only on the great interest in the subject, but also because for a Severi variety V on an arbitrary projective variety X there are several problems concerning V like non-emptiness, smoothness, irreducibility, dimensional computation as well as enumerative and moduli properties of the family of curves it parametrizes.

On the contrary, in higher dimension only a few results are known. Therefore, in [12] we focused on the next relevant case, from the point of view of Algebraic Geometry: families of nodal curves on smooth, projective threefolds.

In this paper, we show that the correspondence considered in [12], which was introduced as an auxiliary tool for some related problems, reflects deep geometric properties of global sections of rank-two vector bundles on a smooth threefold. We also study some of its intriguing consequences, which were not explored in [12].

To be more precise, let X be a smooth projective threefold and let $\mathcal{F}$ be a rank-two vector bundle on X, which is assumed to be globally generated with general global section s whose zero-locus $V(s)$ is a smooth, irreducible curve $D = D_s$ in X, of geometric genus $g(D) = p_a(D)$.

Take now $\mathbb{P}(H^0(X, \mathcal{F}))$; from our assumptions on $\mathcal{F}$, its general point parametrizes a global section whose zero-locus is a smooth, irreducible curve. This projective space somehow gives a scheme dominating a subvariety in which the curves move.

Given a positive integer $\delta \leq p_a(D)$, it makes sense to consider the locally closed subscheme:

$$\begin{aligned}\mathcal{V}_\delta(\mathcal{F}) :=\{[s] \in \mathbb{P}(H^0(X, \mathcal{F})) \mid C_s := V(s) \subset X & \text{ is irreducible} \\ \text{with only } \delta \text{ nodes as singularities}\};\end{aligned}$$

(cf. formula (1.3)). These are usually called *Severi varieties* of global sections of $\mathcal{F}$ whose zero-loci are irreducible, δ-nodal curves in X, of arithmetic genus $p_a(D)$ and geometric genus $g = p_a(D) - \delta$ (cf. [2], for $X = \mathbb{P}^3$, and [12] in general). This is because such schemes are the natural generalization of the (classical) Severi varieties on smooth, projective surfaces recalled before.

When $\mathcal{V}_\delta(\mathcal{F})$ is not empty then its expected codimension in $\mathbb{P}(H^0(X, \mathcal{F}))$ is δ (see e.g. Proposition 1.4). Thus, one says that a point $[s] \in \mathcal{V}_\delta(\mathcal{F})$ is a *regular point* if it is smooth and such that $\dim_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$ equals the expected one (cf. Definition 1.5). In order to find regularity conditions, we introduced in [12] a cohomological description of the tangent space $T_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$ (cf. Theorem 3.4 in [12]). In Section 2, we shall briefly recall this main result, as well as some of its corollaries, not only for the reader convenience but mainly because it is useful for the present paper. Precisely, we recall:

Theorem 1 (see Theorem 2.1 and Proposition 2.3)

Let X be a smooth projective threefold. Let $\mathcal{F}$ be a globally generated rank-two vector bundle on X and δ a positive integer. Consider $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and let $C = V(s) \subset X$. Denote by Σ the set of nodes of C. Let

$$\mathcal{P} := \mathbb{P}_X(\mathcal{F}) \xrightarrow{\pi} X$$

be the projective space bundle together with its natural projection π on X and denote by $\mathcal{O}_\mathcal{P}(1)$ its tautological line bundle. Then:

(i) $[s] \in \mathcal{V}_\delta(\mathcal{F})$ *corresponds to a divisor $G_s \in |\mathcal{O}_\mathcal{P}(1)|$ which contains the δ fibres $L_{p_i} = \pi^{-1}(p_i) \subset \mathcal{P}$, where $p_i \in \Sigma$ for $1 \le i \le \delta$.*

Furthermore, there exists a zero-dimensional subscheme $\Sigma^1 \subset G_s$ of length δ, which is a set of δ rational double points of G_s and each fibre L_{p_i} contains only one of the points of Σ^1, for $1 \le i \le \delta$.

(ii) Denote by $\mathcal{J}_{\Sigma^1/\mathcal{P}}$ the ideal sheaf of Σ^1 in $\mathcal{P}$. The subsheaf of $\mathcal{F}$ defined by

$$\mathcal{F}^\Sigma := \pi_*\big(\mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)\big) \tag{0.1}$$

is such that

$$\frac{H^0(X,\mathcal{F}^\Sigma)}{\langle s\rangle} \cong T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \subset T_{[s]}\big(\mathbb{P}(H^0(\mathcal{F}))\big) \cong \frac{H^0(X,\mathcal{F})}{\langle s\rangle}$$

i.e. global sections of $\mathcal{F}^\Sigma$ (modulo the one-dimensional subspace $\langle s\rangle$) parametrize equisingular *first-order deformations of $[s]$. Thus, for $\epsilon \in \mathbb{C}[T]/(T^2)$ s.t. $\epsilon^2 = 0$, we have:*

$$s + \epsilon s' \in T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \Leftrightarrow \ s' \in H^0(X,\mathcal{F}^\Sigma) \Leftrightarrow G_{s'} \in |\mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)|,$$

where $G_{s'}$ is the divisor in $\mathcal{P}$ corresponding to s'.

For brevity sake, if $\Lambda := \bigcup_{i=1}^{\delta} L_{p_i} = \pi^{-1}(\Sigma)$, we shall say that the elements $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and $G_s \in |\mathcal{J}_{\Lambda/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)|$ as above form a (s, G_s)-*Severi correspondence* (cf. Definition 2.2). The terminology of *Severi correspondence* will be further motivated by other results in the last sections (cf. Definition 4.1, Theorems 3.1 and 5.1 and Corollary 4.19).

The aim of this paper is twofold: first we study in details the geometric meaning of $T_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$ as well as of the (s, G_s)-Severi correspondence. We determine some interesting consequences of this approach, which have not been explored in [12]. We also describe several interesting connections with families of other singular geometric "objects" related to X. In particular, we show that the local analytical computations introduced in [2] are equivalent, via the (s, G_s)-correspondence, to those using the divisorial approach of [12] (cf. Remarks 3.11, 3.21 and Propositions 3.14, 3.19).

On the other hand, we also show that the (s, G_s)-Severi correspondence resides in other deep geometric reasons (cf. Theorems 4.5 and 5.1). We believe that this could be a useful approach for several related problems on higher dimensional varieties; this will be the subject of a future research.

The following first result of the paper gives a converse to the procedure introduced in [12]. Indeed, we show:

Theorem 2 (cf. Theorem 3.1)

Let X be a smooth projective threefold. Let $\mathcal{F}$ be a globally generated rank-two vector bundle on X.

Let $\mathcal{P} := \mathbb{P}_X(\mathcal{F})$ be the projective space bundle, $\mathcal{O}_{\mathcal{P}}(1)$ its tautological line bundle and π the natural projection onto X. Let $G_s \in |\mathcal{O}_{\mathcal{P}}(1)|$ be a divisor and let $s \in H^0(X, \mathcal{F})$ be the global section corresponding to G_s. Let $C := V(s)$ and assume that C is a curve (not necessarily irreducible) on X. Thus:

(i) G_s is singular at a point $p^1 \in \mathcal{P}$ if, and only if, C is singular at the point $p \in C$, which is uniquely determined by the fact that $p^1 \in L_p = \pi^{-1}(p)$.

(ii) In particular, p is a node for C if, and only if, p^1 is a rational double point for G_s.

This is a basic tool for the results contained in Sections 4 and 5.

A related important aspect of the (s, G_s)-Severi correspondence is that we can determine several equivalent geometric interpretations of first-order deformations given by sections in $H^0(X, \mathcal{F}^{\Sigma})$ via Theorems 1 and 2. Indeed, by using our divisorial approach, we prove:

Proposition 1 (cf. Proposition 3.14 and Proposition 3.19)

Let X be a smooth projective threefold, $\mathcal{F}$ a globally generated rank-two vector bundle on X and $\mathcal{L} = c_1(\mathcal{F})$. Let δ be a positive integer, $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and $C = V(s)$ be the corresponding irreducible, nodal curve in X. Denote by Σ the set of nodes of C. Let $\mathcal{F}^{\Sigma}$ be as in (0.1). Then, the following conditions are equivalent:

(i) $s' \in H^0(X, \mathcal{F}^{\Sigma}) \setminus \langle s \rangle$;

(ii) $V(s \wedge s') \subset X$ is a surface which contains C and which is singular along Σ;

(iii) the divisor $G_{s'}$ passes through Σ^1;

(iv) the surface $\mathcal{S}_{s,s'} := G_s \cap G_{s'} \subset \mathcal{P}$ is singular along Σ^1.

Interpretation (ii) above of $T_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$ has already been considered for $X = \mathbb{P}^3$ and via a different approach in Proposition 2.3 in [2]. This in particular shows that the local computations introduced in [2] are equivalent, via the (s, G_s)-Severi correspondence, to the local computations on $\mathcal{P}$. Furthermore, the several distinct characterizations of tangent vectors to $\mathcal{V}_\delta(\mathcal{F})$ at $[s]$ given by Proposition 1 are consistent with the equivalent conditions of regularity for $\mathcal{V}_\delta(\mathcal{F})$ determined in [12] (cf. Corollary 2.15 and Remarks 3.22, 3.25).

What stated up to now suggests that the equivalence given by Theorems 1, 2 and by Proposition 1 more deeply resides in the fact that the theory of Severi varieties of nodal sections $\mathcal{V}_\delta(\mathcal{F})$ on X can be rephrased in terms of "classical" Severi varieties of irreducible, singular divisors on $\mathbb{P}_X(\mathcal{F})$. Indeed, denote by

$$\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1)) := \{G_s \in |\mathcal{O}_{\mathcal{P}}(1)| \text{ s.t. } [s] \in \mathcal{V}_\delta(\mathcal{F})\}$$

the schemes parametrizing families of expected codimension δ in $|\mathcal{O}_{\mathcal{P}}(1)|$, whose elements are irreducible divisors with only δ rational double points as singularities and which correspond to irreducible, nodal curves on X given by zero-loci of global sections of $\mathcal{F}$. For brevity sake, these are called $\mathcal{P}_\delta(\mathcal{F})$-*Severi varieties* (cf. Definition 4.1 and formula (4.3)).

We remark that, by the (s, G_s)-Severi correspondence, if $[s] \in \mathcal{V}_\delta(\mathcal{F})$ then the corresponding divisor G_s is irreducible. Conversely, given an arbitrary irreducible divisor $G_s \subset \mathcal{P}$ with δ-rational double points as the only singularities, take s the corresponding global section of $\mathcal{F}$; even if we assume that $C = V(s)$ is of codimension two in X and with only δ nodes as singularities, it does not follow that C is necessarily irreducible. We discuss some examples in Remark 3.12 which show that the (s, G_s)-Severi correspondence is not one-to-one and which motivate the above definition of $\mathcal{P}_\delta(\mathcal{F})$-*Severi varieties.*

We first prove:

Theorem 3 (cf. Theorem 4.5 and Corollary 4.19)

Let $[G_s] \in \mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ on $\mathcal{P}$ and let Σ^1 be the zero-dimensional scheme of the δ-rational double points of $G_s \subset \mathcal{P}$. Then

$$T_{[G_s]}\big(\mathcal{R}_\delta\big(\mathcal{O}_{\mathcal{P}}(1)\big)\big) \cong \frac{H^0(\mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1))}{\langle G_s \rangle}.$$

In particular,

$$[G_s] \in \mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1)) \text{ is a regular point} \Leftrightarrow [s] \in \mathcal{V}_\delta(\mathcal{F}) \text{ is a regular point.}$$

Finally, we deduce regularity results for $\mathcal{P}_\delta(\mathcal{F})$-Severi varieties $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ on $\mathcal{P}$; indeed, thanks to the (s, G_s)-Severi correspondence and to regularity results of $\mathcal{V}_\delta(\mathcal{F})$ in [12], we state:

Theorem 4 (cf. Theorem 5.1)

Let X be a smooth projective threefold, $\mathcal{E}$ be a globally generated rank-two vector bundle on X, M be a very ample line bundle on X and $k \geq 0$ and $\delta > 0$ be integers. Let $\mathcal{P} := \mathbb{P}_X(\mathcal{E} \otimes M^{\otimes k})$ and $\mathcal{O}_{\mathcal{P}}(1)$ be its tautological line bundle. If

$$(*) \qquad \delta \leq k + 1,$$

then $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ on $\mathcal{P}$ are regular at each point.

The upper-bounds in $(*)$ are shown to be almost sharp (cf. Remark 5.3).

The above result highlights once more the fundamental role of the (s, G_S)-Severi correspondence. Indeed, if one considers the $\mathcal{P}_\delta(\mathcal{F})$-Severi varieties independently from the corresponding varieties $\mathcal{V}_\delta(\mathcal{F})$ on X, the regularity condition for a point of $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ is equivalent to the separation of suitable zero-dimensional schemes by the linear system $|\mathcal{O}_{\mathcal{P}}(1)|$ on the fourfold $\mathcal{P}$ (cf. Corollary 4.19). In general, it is well-known how difficult is to establish when a linear system separates points in projective varieties of dimension greater than or equal to three (cf. e.g. [1], [9] and [17]). In some cases, some separation results can be found by using technical tools like *multiplier ideals* as well as the Nadel and the Kawamata-Viehweg vanishing theorems (see, e.g. [8], for an overview). In our situation, thanks to the correspondence between $\mathcal{V}_\delta(\mathcal{F})$ on X and $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ on $\mathcal{P}$, we deduce regularity conditions for $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ from those already obtained for $\mathcal{V}_\delta(\mathcal{F})$ in [12].

The paper consists of five sections. Section 1 contains some general terminology and notation.

In Section 2 we briefly remind some fundamental definitions and results of [12], not only for the reader convenience but mainly because some tools are frequently used in the whole paper. The aim of Section 3 is to study in more details the (s, G_s)-Severi correspondence. We consider several important geometric consequences of this correspondence (cf. e.g. Theorem 3.1, Propositions 3.14, 3.19) as well as the equivalence of some of these consequences with the approach used in [2].

In Section 4 we focus on $\mathcal{P}_\delta(\mathcal{F})$-Severi varieties; we give a description of tangent spaces at points of such schemes as well as we find conditions for their regularity (cf. Theorem 4.5 and Corollary 4.19). Section 5 contains some almost-sharp upper-bounds on δ which imply the regularity of $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ on $\mathcal{P}$.

Acknowledgments: Part of this paper is related to [12], which was prepared during my permanence at the Department of Mathematics of the University of Illinois at Chicago. Therefore, my deepest gratitude goes to L. Ein, for having suggested me to approach this pioneering area. My very special thanks go also to GNSAGA-INdAM and to V. Barucci, A. F. Lopez and E. Sernesi for their confidence and their support during my period in U.S.A.

I am grateful to L. Caporaso and to the organizers of the Workshop *"Global geometry of algebraic varieties"* - Madrid, December 2002, since their invitations to give talks have given "life" to these new results.

I am indebted to L. Chiantini and to C. Ciliberto for many remarks on the subject and for having always stimulated me to investigate more in this research area.

Finally, I warmly thank the referee for the advice and remarks on a better reorganization of the first version of this manuscript.

1. Notation and Preliminaries

We work in the category of algebraic $\mathbb{C}$-schemes. Y is a *m-fold* if it is a reduced, irreducible and non-singular scheme of finite type and of dimension m. If $m = 1$, then Y is a (smooth) *curve*; $m = 2, 3$ and 4 are the cases of a (non-singular) *surface*, *threefold* and *fourfold*, respectively. If Z is a closed subscheme of a scheme Y, $\mathcal{J}_{Z/Y}$ denotes the *ideal sheaf* of Z in Y, $\mathcal{N}_{Z/Y}$ the *normal sheaf* of Z in Y whereas $\mathcal{N}^\vee_{Z/Y} \cong \mathcal{J}_{Z/Y}/\mathcal{J}^2_{Z/Y}$ is the *conormal sheaf* of Z in Y. As usual, $h^i(Y, -) := \dim H^i(Y, -)$.

Given Y a projective scheme, ω_Y denotes its dualizing sheaf. When Y is a smooth variety, then ω_Y coincides with its canonical bundle and K_Y denotes a canonical divisor s.t. $\omega_Y \cong \mathcal{O}_Y(K_Y)$; furthermore, $\mathcal{T}_Y$ denotes its tangent bundle.

If D is a reduced, irreducible curve, $p_a(D) = h^1(\mathcal{O}_D)$ denotes its *arithmetic genus*, whereas $g(D) = p_g(D)$ denotes its *geometric genus*, the arithmetic genus of its normalization.

Let Y be a projective m-fold and $\mathcal{E}$ be a rank-r vector bundle on Y; $c_i(\mathcal{E})$ denotes the i^{th}-*Chern class* of $\mathcal{E}$, $1 \leq i \leq r$. As in [16] - Section II.7 - $\mathbb{P}_Y(\mathcal{E})$ denotes the *projective space bundle* on Y, defined as $Proj(Sym(\mathcal{E}))$. There is a surjection $\pi^*(\mathcal{E}) \to \mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$, where $\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$ is the *tautological line bundle* on $\mathbb{P}_Y(\mathcal{E})$ and where $\pi : \mathbb{P}_Y(\mathcal{E}) \to Y$ is the natural projection morphism.

For non reminded terminology, the reader is referred to [3], [13] and [16]. We now briefly recall some definitions and results which will be frequently used in what follows.

Let X be a smooth projective threefold and $\mathcal{F}$ a rank-two vector bundle on X. If $\mathcal{F}$ is globally generated on X, it is not restrictive if from now on we assume that the zero-locus $V(s)$ of its general global section s is a smooth, irreducible curve $D = D_s$ in X (for details, see [12]; for general motivations and backgrounds, the reader is referred to e.g. [19] and to [24], Chapter IV).

From now on, denote by $\mathcal{L} \in Pic(X)$ the line bundle given by $c_1(\mathcal{F})$. Thus, by the *Koszul sequence* of $(\mathcal{F}, s)$:

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{J}_D \otimes \mathcal{L} \to 0, \tag{1.1}$$

we compute the geometric genus of $D = V(s)$ in terms of the invariants of $\mathcal{F}$ and of X. Precisely

$$2g(D) - 2 = 2p_a(D) - 2 = \deg(\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_D). \tag{1.2}$$

Thus, if e.g. $X = \mathbb{P}^3$ and if we put $c_i = c_i(\mathcal{F}) \in \mathbb{Z}$, we have

$$\deg(D) = c_2 \text{ and } g(D) = p_a(D) = \frac{1}{2}(c_2(c_1 - 4)) + 1,$$

i.e. D is subcanonical of level $(c_1 - 4)$.

Take now $\mathbb{P}(H^0(X, \mathcal{F}))$; from our assumptions on $\mathcal{F}$, the general point of this projective space parametrizes a global section whose zero-locus is a smooth, irreducible curve in X. Given a positive integer $\delta \leq p_a(D)$, one consider the subset

$$\begin{aligned}\mathcal{V}_\delta(\mathcal{F}) :=& \{[s] \in \mathbb{P}(H^0(X, \mathcal{F})) \mid C_s := V(s) \subset X \text{ is irreducible} \\ & \text{with only } \delta \text{ nodes as singularities}\}\,;\end{aligned} \tag{1.3}$$

therefore, any element of $\mathcal{V}_\delta(\mathcal{F})$ determines a curve in X whose arithmetic genus $p_a(C_s)$ is given by (1.2) and whose geometric genus is $g = p_a(C_s) - \delta$. We recall that $\mathcal{V}_\delta(\mathcal{F})$ is a locally closed subscheme of the projective space $\mathbb{P}(H^0(X, \mathcal{F}))$; it is usually called the *Severi variety* of global sections of $\mathcal{F}$ whose zero-loci are irreducible, δ-nodal curves in X (cf. [2], for $X = \mathbb{P}^3$, and [12] in general). This is because such schemes are the natural generalization of the (classical) Severi varieties of irreducible and δ-nodal curves in linear systems on smooth, projective surfaces (see [5], [4], [7], [11], [14], [15], [21], [22] and [23], just to mention a few).

For brevity sake, we shall usually refer to $\mathcal{V}_\delta(\mathcal{F})$ as the *Severi variety of irreducible, δ-nodal sections* of $\mathcal{F}$ on X.

First possible questions on such Severi varieties are about their dimensions as well as their smoothness properties. A preliminary estimate is given by the following result:

Proposition 1.4

Let X be a smooth projective threefold, $\mathcal{F}$ a globally generated rank-two vector bundle on X and δ a positive integer. Then

$$\operatorname{expdim}(\mathcal{V}_\delta(\mathcal{F})) = \begin{cases} h^0(X,\mathcal{F}) - 1 - \delta, & \text{if } \delta \leq h^0(X,\mathcal{F}) - 1 = \dim\left(\mathbb{P}(H^0(\mathcal{F}))\right), \\ -1, & \text{if } \delta \geq h^0(X,\mathcal{F}). \end{cases}$$

Proof. See Proposition 2.10 in [12]. $\square$

Assumption 1. From now on, given X and $\mathcal{F}$ as in Proposition 1.4, we shall always assume $\mathcal{V}_\delta(\mathcal{F}) \neq \emptyset$. We write $[s] \in \mathcal{V}_\delta(\mathcal{F})$ to intend that the global section $s \in H^0(X,\mathcal{F})$ determines the corresponding point $[s]$ of the scheme $\mathcal{V}_\delta(\mathcal{F})$. We simply denote by C (instead of C_s) its zero-locus, when it is clear from the context that we focus on s. We finally consider $\delta \leq \min\{h^0(X,\mathcal{F}) - 1,\ p_a(C)\}$, the latter is because we want $C = V(s)$ to be irreducible, for any $[s] \in \mathcal{V}_\delta(\mathcal{F})$.

By Proposition 1.4, it is natural to state the following:

DEFINITION 1.5. Let $[s] \in \mathcal{V}_\delta(\mathcal{F})$, with $\delta \leq \min\{h^0(X,\mathcal{F}) - 1,\ p_a(C)\}$. Then $[s]$ is said to be a *regular point* of $\mathcal{V}_\delta(\mathcal{F})$ if:

(i) $[s] \in \mathcal{V}_\delta(\mathcal{F})$ is a smooth point, and
(ii) $\dim_{[s]}(\mathcal{V}_\delta(\mathcal{F})) = \operatorname{expdim}(\mathcal{V}_\delta(\mathcal{F})) = \dim\left(\mathbb{P}(H^0(X,\mathcal{F}))\right) - \delta$.

$\mathcal{V}_\delta(\mathcal{F})$ is said to be *regular* if it is regular at each point.

In [12] we presented a cohomological description of the tangent space $T_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$ which allowed us to find several sufficient conditions for the regularity of Severi varieties $\mathcal{V}_\delta(\mathcal{F})$ on X (cf. Theorems 4.5, 5.9, 5.25, 5.28 and 5.36 in [12]).

One of the aim of this paper is to study in more details the deep geometric meaning of this cohomological description of $T_{[s]}(\mathcal{V}_\delta(\mathcal{F}))$ and its several connections (not investigated in [12]) with families of other singular geometric objects related to X and to $\mathcal{F}$.

To do this, we first have to recall some results which are the starting point of our analysis.

2. The (s, G_s)-Severi correspondence

In this section we want to briefly recall the correspondence given in [12] between elements of $\mathcal{V}_\delta(\mathcal{F})$ on X and suitable singular divisors on the projective space bundle

$\mathcal{P} := \mathbb{P}_X(\mathcal{F})$, which is a fourfold ruled over X. This will be called the (s, G_s)-*Severi correspondence*, as in Definition 2.2.

From now on, with conditions as in Assumption 1, let $[s] \in \mathcal{V}_\delta(\mathcal{F})$. Then, in [12] we proved:

Theorem 2.1 (cf. Theorem 3.4 (i) in [12])

Let X be a smooth projective threefold. Let $\mathcal{F}$ be a globally generated rank-two vector bundle on X and δ a positive integer. Consider $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and let $C = V(s) \subset X$. Denote by Σ the set of nodes of C.
Let

$$\mathcal{P} := \mathbb{P}_X(\mathcal{F}) \xrightarrow{\pi} X$$

be the projective space bundle together with its natural projection π on X and denote by $\mathcal{O}_{\mathcal{P}}(1)$ its tautological line bundle.

Then, $[s] \in \mathcal{V}_\delta(\mathcal{F})$ corresponds to a divisor $G_s \in |\mathcal{O}_{\mathcal{P}}(1)|$ which contains the δ fibres $L_{p_i} = \pi^{-1}(p_i) \subset \mathcal{P}$, where $p_i \in \Sigma$ for $1 \leq i \leq \delta$.

Furthermore, there exists a zero-dimensional subscheme $\Sigma^1 \subset G_s$ of length δ, which is a set of δ rational double points of G_s and each fibre L_{p_i} contains only one of the points of Σ^1, for $1 \leq i \leq \delta$.

Proof. For complete details, the reader is referred to the proof of Theorem 3.4 (i) in [12]. □

For brevity sake, we give the following:

DEFINITION 2.2. With notation and assumptions as in Theorem 2.1, let $\Lambda = \pi^{-1}(\Sigma) = \bigcup_{i=1}^{\delta} L_{p_i}$. Then, the elements $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and $G_s \in |\mathcal{J}_{\Lambda/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1)|$ are said to form a (s, G_s)-*Severi correspondence*.

The terminology of "Severi correspondence" will be further motivated by Definition 4.1, Theorems 3.1 and 5.1 and Corollary 4.19.

Another result of [12] which is useful to remind is the following:

Proposition 2.3 (cf. Theorem 3.4 (ii) in [12])

With assumptions and notation as in Theorem 2.1, denote by $\mathcal{J}_{\Sigma^1/\mathcal{P}}$ the ideal sheaf of Σ^1 in $\mathcal{P}$. Consider the subsheaf of $\mathcal{F}$ defined by

$$\mathcal{F}^{\Sigma} := \pi_*(\mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1)). \tag{2.4}$$

Then,

$$\frac{H^0(X, \mathcal{F}^{\Sigma})}{\langle s \rangle} \cong T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \subset T_{[s]}\big(\mathbb{P}(H^0(\mathcal{F}))\big) \cong \frac{H^0(X, \mathcal{F})}{\langle s \rangle}, \tag{2.5}$$

i.e. global sections of $\mathcal{F}^{\Sigma}$ (modulo the one-dimensional subspace $\langle s \rangle$) parametrize equisingular *first-order deformations of $[s]$. In particular, for $\epsilon \in \mathbb{C}[T]/(T^2)$ s.t. $\epsilon^2 = 0$, we have:*

$$s + \epsilon s' \in T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \Leftrightarrow \; s' \in H^0(X, \mathcal{F}^{\Sigma}) \Leftrightarrow G_{s'} \in |\mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1)|, \tag{2.6}$$

where $G_{s'}$ is the divisor in $\mathcal{P}$ corresponding to s'.

Proof. The reader is referred to the proof of Theorem 3.4 (ii) in [12]. □

Remark 2.7. As in [12], one can show that $\mathcal{F}^{\Sigma}$ fits in the exact diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & 0 \to & \mathcal{J}_{C/X} \otimes \mathcal{F} & \stackrel{\cong}{\to} & \mathcal{J}_{C/X} \otimes \mathcal{F} & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
(2.8) & 0 \to & \mathcal{F}^{\Sigma} & \to & \mathcal{F} & \to & \mathcal{O}_{\Sigma} & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow^{\cong} & & \\
 & 0 \to & \mathcal{N}'_C & \to & \mathcal{F}|_C & \to & T^1_C & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where T^1_C is the *first cotangent sheaf* of C (for details, see [20]) and where $\mathcal{N}'_C$ is the *equisingular sheaf* defined as the kernel of the natural surjection

$$0 \to \mathcal{N}'_C \to \mathcal{N}_{C/X} \to T^1_C \to 0, \tag{2.9}$$

(see, for example, [22]). Recall also that, since $C = V(s)$ and since it is nodal, then $\mathcal{N}_{C/X} \cong \mathcal{F}|_C$ and $T^1_C \cong \mathcal{O}_{\Sigma}$.

By the second row of (2.8) one can consider the map

$$H^0(X, \mathcal{F}) \xrightarrow{\mu_X} H^0(X, \mathcal{O}_{\Sigma}), \tag{2.10}$$

which is not defined by evaluating the global sections of $\mathcal{F}$ at Σ (indeed $\mathcal{F}^{\Sigma}$ has rank one at each node). Its geometric meaning is given by the local description of the exact sequence

$$0 \to \mathcal{F}^{\Sigma} \to \mathcal{F} \to \mathcal{O}_{\Sigma} \to 0. \tag{2.11}$$

If $p \in \Sigma$, take $U_p \subset X$ an analytical neighbourhood of p with local coordinates $\underline{x} = (x_1, x_2, x_3)$ such that $\underline{x}(p) = (0,0,0)$. If $s \in H^0(X, \mathcal{F})$ is such that $s|_{U_p} = (f_1, f_2)$ then, as in [2] for $X = \mathbb{P}^3$, one can consider its Jacobian map

$$\mathcal{T}_{U_p}|_C \stackrel{J(s|_{U_p})}{\longrightarrow} \mathcal{N}_{C/U_p} \to T^1_C, \tag{2.12}$$

which is given by:

$$J(s|_{U_p}) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{pmatrix}. \tag{2.13}$$

Since in our case $[s] \in \mathcal{V}_\delta(\mathcal{F})$, by (2.13) one easily sees that μ_X is the composition of the evaluation at p of global sections of $\mathcal{F}$ followed by the projection

$$\mathbb{C}^2_{(p)} \xrightarrow{\pi_1} \mathbb{C}_{(p)},$$

where $\mathbb{C}^2_{(p)} \cong \mathcal{F} \otimes \mathcal{O}_p$, $\mathbb{C}_{(p)} \cong T^1_{C,p}$ and $\pi_1((x,y)) = x$ (for more details, cf. § 3 in [12]).

On the other hand, by the (s, G_s)-Severi correspondence, one can also consider the standard evaluation map at Σ^1 of tautological divisors on $\mathcal{P}$, which will be denoted by

$$H^0(\mathcal{P}, \mathcal{O}_{\mathcal{P}}(1)) \xrightarrow{\rho_{\mathcal{P}}} H^0(\mathcal{P}, \mathcal{O}_{\Sigma^1}). \tag{2.14}$$

In [12], we deduced the following result which will play a fundamental role in § 5 (cf. Corollary 4.19 and Theorem 5.1).

Corollary 2.15 (cf. Corollary 3.9 in [12])

From (2.8), it follows that

$$\begin{aligned} [s] \in \mathcal{V}_\delta(\mathcal{F}) \text{ is regular} &\Leftrightarrow H^0(X, \mathcal{F}) \overset{\mu_X}{\twoheadrightarrow} H^0(X, \mathcal{O}_\Sigma) \\ &\Leftrightarrow H^0(\mathcal{P}, \mathcal{O}_{\mathcal{P}}(1)) \overset{\rho_{\mathcal{P}}}{\twoheadrightarrow} H^0(\mathcal{P}, \mathcal{O}_{\Sigma^1}). \end{aligned} \tag{2.16}$$

Proof. The equivalence of μ_X and $\rho_{\mathcal{P}}$ easily follows from the (s, G_s)-Severi correspondence (see also § 3 in [12]). Formula (2.16) directly follows from Proposition 1.4, Theorem 2.1 and Proposition 2.3. □

3. Connection among various singular subschemes of X and of $\mathcal{P}$

The aim of this section is to study in more details the (s, G_s)-Severi correspondence of Definition 2.2, introduced in [12] and briefly recalled in the previous section. We consider several important geometric consequences of this correspondence. In particular, we show that the local analytical computations introduced in [2] are equivalent via the (s, G_s)-correspondence to those using the divisorial approach in [12] (cf. Remarks 3.11, 3.21 and Propositions 3.14, 3.19).

We start with the following result, which determines a converse of the approach introduced in [12]. Indeed, we more generally prove:

Theorem 3.1

Let X be a smooth projective threefold. Let $\mathcal{F}$ be a globally generated rank-two vector bundle on X.

Let $\mathcal{P} := \mathbb{P}_X(\mathcal{F})$ be the projective space bundle, $\mathcal{O}_{\mathcal{P}}(1)$ its tautological line bundle and π the natural projection onto X. Let $G_s \in |\mathcal{O}_{\mathcal{P}}(1)|$ be a divisor and let $s \in H^0(X, \mathcal{F})$ be the global section corresponding to G_s. Let $C := V(s)$ and assume that C is a curve (not necessarily irreducible) on X. Thus:

(i) G_s is singular at a point $p^1 \in \mathcal{P}$ if, and only if, C is singular at the point $p \in C$, which is uniquely determined by the fact that $p^1 \in L_p = \pi^{-1}(p)$.
(ii) In particular, p is a node for C if, and only if, p^1 is a rational double point for G_s.

Proof. (i) Let $p^1 \in G_s$ and let $p \in X$ such that $p^1 \in L_p = \pi^{-1}(p) \subset \mathcal{P}$. Let $U = U_p$ be an analytical neighbourhood of p in X where $\mathcal{F}$ trivializes and whose local coordinates are $\underline{x} = (x_1, x_2, x_3)$. If $s|_U = (f_1, f_2)$, by definition of projective space bundle, the local equation of G_s in $\pi^{-1}(U)$ is

$$F(x_1, x_2, x_3, u, v) := uf_1 + vf_2, \tag{3.2}$$

where $[u, v]$ are local homogeneous coordinates on the fibres over U. Therefore, $p^1 \in \pi^{-1}(U)$ is singular for G_s if, and only if, there exists a solution of

$$F = \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial v} = 0. \tag{3.3}$$

Observe that (3.3) gives:

$$\begin{aligned} uf_1 + vf_2 = \frac{\partial f_1}{\partial x_1}u + \frac{\partial f_2}{\partial x_1}v &= \frac{\partial f_1}{\partial x_2}u + \frac{\partial f_2}{\partial x_2}v \\ &= \frac{\partial f_1}{\partial x_3}u + \frac{\partial f_2}{\partial x_3}v = f_1 = f_2 = 0. \end{aligned} \tag{3.4}$$

The last two equations of (3.4) imply that a singular point of G_s must be on the π-fibre over a point of the locus $C = V(s) \subset X$. This means that $p \in U \cap C$; let $L = L_p$ be its fibre.

We can restrict the system (3.4) to L (by a little abuse of notation, we shall always denote by $[u, v]$ the homogeneous coordinates on L). We thus get:

$$\frac{\partial f_1}{\partial x_1}(p)u + \frac{\partial f_2}{\partial x_1}(p)v = \frac{\partial f_1}{\partial x_2}(p)u + \frac{\partial f_2}{\partial x_2}(p)v = \frac{\partial f_1}{\partial x_3}(p)u + \frac{\partial f_2}{\partial x_3}(p)v = 0. \tag{3.5}$$

Therefore, there exists a solution $[u, v] \in L \cong \mathbb{P}^1$ if, and only if, the system (3.5) has rank less than or equal to one. This is equivalent to saying that

$$\Big(\frac{\partial f_1}{\partial x_1}(p), \frac{\partial f_1}{\partial x_2}(p), \frac{\partial f_1}{\partial x_3}(p)\Big) = \lambda\Big(\frac{\partial f_2}{\partial x_1}(p), \frac{\partial f_2}{\partial x_2}(p), \frac{\partial f_2}{\partial x_3}(p)\Big), \tag{3.6}$$

for some $\lambda \in \mathbb{C}^*$, i.e.

$$\mathrm{rank}(J(s|_U)(p)) \leq 1, \tag{3.7}$$

with $J(s|_U)$ as in (2.13). This is equivalent to the fact that $p \in Sing(C)$. Furthermore, by the above computations and by the definition of projective space bundle, one sees that the point $p^1 \in L_p$ has homogeneous coordinates:

$$\begin{aligned} [u, v] = \Big[-\frac{\partial f_2}{\partial x_1}(p), \frac{\partial f_1}{\partial x_1}(p)\Big] &= \Big[-\frac{\partial f_2}{\partial x_2}(p), \frac{\partial f_1}{\partial x_2}(p)\Big] \\ &= \Big[-\frac{\partial f_2}{\partial x_3}(p), \frac{\partial f_1}{\partial x_3}(p)\Big] = [-\lambda, 1] \end{aligned} \tag{3.8}$$

(when they make sense).

(ii) ($\Rightarrow$) As in Theorem 2.1 (cf. Theorem 3.4 (i) in [12]), the claim follows from the fact that locally analytically $s|_U = (x_1x_2, x_3)$, since a node is a planar singularity.
($\Leftarrow$) One can use part (i) above and the fact that, in suitable local analytical coordinates, a rational double point of a threefold is always locally given by $\{x_1x_2+x_3t=0\}$, where (x_1, x_2, x_3, t) are coordinates in $\mathbb{A}^4$. Thus, the local equation of G_s in $\pi^{-1}(U) = U \times \mathbb{P}^1$ is given by

$$ux_1x_2 + vx_3 = 0, \tag{3.9}$$

where $[u, v]$ are the homogeneous coordinates of the fibres over U and $t = \frac{v}{u}$. Then s is locally analytically given by

$$s|_{U_p} = (x_1x_2,\ x_3), \tag{3.10}$$

so one can conclude. □

Remark 3.11. Observe first that, from (3.2), it easily follows that the multiplicity of G_s at the point p^1 along the π-fibre L_p increases when $C = V(s)$ has multiplicity worse than two at p.

When, in particular, p is a node one can give some geometrical characterizations of the singular point $p^1 \in L_p$ in terms of s. Indeed, since p is a node then $C = V(s)$ is a local complete intersection so $\mathcal{N}_{C/X}$ is locally free of rank two on C. By (2.12) and (2.13), one observes that in this case $Im(J(s|_U))$ does not generate the whole vector space $V := (\mathcal{O}_{C,p}/\underline{m}_p)^{\oplus 2}$ given by elements in $\mathcal{N}_{C/X,p}$ not vanishing at p; in fact, if $s|_U := (f_1, f_2)$, then $Im(J(s|_U))$ generates the one-dimensional subspace:

$$W := \left\{ v \in \mathbb{C}^2 \mid v \in \left\langle \frac{\partial f_1}{\partial x_1}(p), \frac{\partial f_2}{\partial x_1}(p) \right\rangle \right\},$$

where
$$\left\langle \frac{\partial f_1}{\partial x_1}(p), \frac{\partial f_2}{\partial x_1}(p) \right\rangle = \left\langle \frac{\partial f_1}{\partial x_2}(p), \frac{\partial f_2}{\partial x_2}(p) \right\rangle = \left\langle \frac{\partial f_1}{\partial x_3}(p), \frac{\partial f_2}{\partial x_3}(p) \right\rangle.$$

By definition of $\mathbb{P}_X(\mathcal{F})$, the points of the fibre L_p are in one-to-one correspondence with the one-dimensional quotients of V. In our case, the singular point p^1 corresponds to the quotient V/W which exactly gives (3.8). The singularity of G_s depends on the fact that the tangent planes at p to the (local) surfaces given by $V(f_1)$ and $V(f_2)$ in U, respectively, are not transverse. In fact, these tangent planes coincide, as it follows from (3.6), so that the curve section $C = V(s)$ has not a unique tangent line.

To conclude, observe that the singular points of G_s move on $\mathcal{P}$ as $[s]$ moves in $\mathcal{V}_\delta(\mathcal{F})$. When $[s]$, $[s'] \in \mathcal{V}_\delta(\mathcal{F})$ are such that $C = V(s)$ and $C' = V(s')$ have a node at the same point $p \in X$, then the singular points p^1 and q^1 of G_s and $G_{s'}$, respectively, are points on the same fibre L_p; in the other case, p^1 and q^1 belong to distinct π-fibres.

Remark 3.12. It is important to observe that the (s, G_s)-Severi correspondence is not a one-to-one correspondence. Indeed, for a given $[s] \in \mathcal{V}_\delta(\mathcal{F})$ the corresponding G_s is irreducible. Conversely, given an arbitrary irreducible divisor $G_s \subset \mathcal{P}$ with δ-rational double points as the only singularities, take s the corresponding global section of $\mathcal{F}$. Even if we assume that $C = V(s)$ is in codimension two in X and with only δ nodes as singularities, it is not true that C is necessarily irreducible.

Indeed, take $X = \mathbb{P}^3$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(2)$. Take $Q_1,\ Q_2 \subset \mathbb{P}^3$ two smooth quadrics given by quadratic polynomials q_i, $1 \leq i \leq 2$, respectively. We can choose the q_i's in such a way that $Q_1 \cap Q_2$ is a divisor $C = \Gamma + L$ on e.g. Q_1, where Γ is a twisted cubic of type $(2,1)$ on Q_1 whereas L is of type $(0,1)$ on Q_1. Since $s = (q_1, q_2) \in H^0(\mathbb{P}^3, \mathcal{F})$ and since $\Gamma L = 2$, then $C = V(s)$ is a 2-nodal curve in $\mathbb{P}^3$ which is reducible. Thus, from (1.3), it follows that $[s] \notin \mathcal{V}_2(\mathcal{F})$.

On the other hand, if we take $\mathcal{P} = Proj(Sym(\mathcal{F}))$, the above section $s \in H^0(X, \mathcal{F})$ comes from a divisor $G_s \in |\mathcal{O}_{\mathcal{P}}(1)|$ which is irreducible. To see this, we first observe that from the exact sequence

$$0 \to \mathcal{O}_{\mathcal{P}}(-1) \to \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{G_s} \to 0$$

it follows that G_s is connected. Since G_s is a divisor in $\mathcal{P}$ with only 2 rational double points as singularities, by dimension count it is also irreducible.

The previous remark motivates Definition 4.1 given in Section 4, where we shall consider the theory of Severi varieties of nodal sections $\mathcal{V}_\delta(\mathcal{F})$ on X in terms of "classical" Severi varieties of irreducible divisors on $\mathbb{P}_X(\mathcal{F})$ with δ-rational double points.

Another important consequence of the (s, G_s)-Severi correspondence is that we can determine several interesting geometric interpretations of first-order deformations given by sections in $H^0(X, \mathcal{F}^\Sigma)$ via the divisorial approach of Theorems 2.1 and 3.1. To start with, let $[s] \in \mathcal{V}_\delta(\mathcal{F})$, $C = V(s)$, $\Sigma = Sing(C)$. Denote by $\mathcal{L} := c_1(\mathcal{F}) \in Pic(X)$.

By (1.1), one has:

$$T_{[s]}(\mathcal{V}_\delta(\mathcal{F})) \subset T_{[s]}\big(\mathbb{P}(H^0(X, \mathcal{F}))\big) \cong \frac{H^0(X, \mathcal{F})}{H^0(X, \mathcal{O}_X)} \hookrightarrow H^0(X, \mathcal{J}_{C/X} \otimes \mathcal{L}). \tag{3.13}$$

Therefore, as proved in Proposition 2.3 in [2] for the case $X = \mathbb{P}^3$ and via another approach, first-order deformations of $[s]$ in the Severi variety $\mathcal{V}_\delta(\mathcal{F})$ can be related to suitable divisors of $|\mathcal{L}|$ on X and containing C.

Precisely, we have:

Proposition 3.14

Let X be a smooth projective threefold, $\mathcal{F}$ a globally generated rank-two vector bundle on X and $\mathcal{L} = c_1(\mathcal{F})$. Let δ be a positive integer, $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and $C = V(s)$ be the corresponding irreducible, nodal curve in X. Denote by Σ the set of nodes of C. Let $\mathcal{F}^\Sigma$ be as in (2.4). Then:

(i) $s' \in H^0(X, \mathcal{F}^\Sigma) \setminus \langle s \rangle$ if, and only if, $V(s \wedge s')$ is a divisor in $|\mathcal{J}_{C/X} \otimes \mathcal{L}|$ which is singular along Σ.

(ii) The singularities of $V(s \wedge s')$ are along Σ and along the (possibly empty) intersection scheme $C \cap V(s')$.

Proof. (i) This point has already been proved in [2] for $X = \mathbb{P}^3$ and via another approach. Here we give our proof which uses the (s, G_s)-Severi correspondence. Let $p \in \Sigma$ be a node of C and let $U = U_p$ be an analytical neighbourhood of X containing p. Let $s' \in H^0(X, \mathcal{F}^{\Sigma}) \setminus \langle s \rangle$ and assume that

$$s|_U = (f_1, f_2),\ s'|_U = (g_1, g_2),\ f_i,\ g_i \in \mathcal{O}_X(U),\ 1 \leq i \leq 2,$$

are the local analytical expression of s and s' in U.
$\Rightarrow$) Denote by $\underline{m}_p$ the maximal ideal of the point p in the stalk $\mathcal{O}_{X,p}$. Since by assumption $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and $p \in \Sigma$, we can assume that the reduction of s in $\mathcal{F} \otimes (\underline{m}_p/\underline{m}_p^2)$ is $(1,0)$. This means that if we consider homogeneous coordinates $[u,v]$ on the π-fibre $L_p \cong \mathbb{P}^1$ over p, the corresponding rational double point p^1 for G_s on L_p has coordinates $[0,1]$ on such a line. Since by assumption $s' \in H^0(\mathcal{F}^{\Sigma}) \setminus \langle s \rangle$, in particular $G_{s'}$ is a divisor distinct from G_s and which passes through $p^1 = [0,1]$. Therefore, we can assume that the reduction of s' in $\mathcal{F} \otimes (\mathcal{O}_{X,p}/\underline{m}_p)$ is $(a, 0)$. If $a = 0$, this means that $G_{s'}$ contains L_p; otherwise, as in (3.9), the local equation of $G_{s'}$ is given by $\{au = 0\}$, so that the intersection point between $G_{s'}$ and L_p is indeed $p^1 = [0,1]$.

In any case, we have that $g_2 \in \underline{m}_p$ and $g_1 = a + j_1$, where $j_1 \in \underline{m}_p$. Analogously, we have that $f_1 \in \underline{m}_p$ and $f_2 \in \underline{m}_p^2$. Therefore,

$$\det \begin{pmatrix} f_1 & f_2 \\ a + j_1 & g_2 \end{pmatrix} = f_1 g_2 - f_2(a + j_1) \in \underline{m}_p^2. \tag{3.15}$$

On the other hand, since $s' \in H^0(X, \mathcal{F}) \setminus \langle s \rangle$, then $V(s \wedge s')$ corresponds to a divisor in $|\mathcal{L}|$ containing $C = V(s)$ whose local equation in U is given by

$$f_1 g_2 - f_2 g_1 = 0. \tag{3.16}$$

By (3.15), it follows that $V(s \wedge s')$ is singular at each node of C.
$\Leftarrow$) Fix (x_1, x_2, x_3) local analytical coordinates in U. Since $p \in C = V(s)$, then

$$\frac{\partial}{\partial x_i}(f_1 g_2 - g_1 f_2)(p) = g_1(p)\frac{\partial f_2}{\partial x_i}(p) - g_2(p)\frac{\partial f_1}{\partial x_i}(p),\ 1 \leq i \leq 3 \tag{3.17}$$

holds. Since $V(s \wedge s')$ is a singular divisor along Σ by assumption, from (3.16) and (3.17) it follows that either s' passes through p or $s'(p)$ is proportional to each pair:

$$\Big(\frac{\partial f_1}{\partial x_i}(p), \frac{\partial f_2}{\partial x_i}(p)\Big),\ 1 \leq i \leq 3. \tag{3.18}$$

In the former case, we have that $G_{s'} \in |\mathcal{J}_{L_p/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1)|$, where L_p is the π-fibre over $p \in \Sigma$; in the latter case, by the very definition of $\mathcal{F}^{\Sigma}$ and by (2.11), (2.12) we have that $G_{s'} \in |\mathcal{J}_{p^1} \otimes \mathcal{O}_{\mathcal{P}}(1)|$, where $p^1 \in \Sigma^1$ is the corresponding point to $p \in \Sigma$. In any case, $G_{s'}$ passes through p^1.

If we globalize this approach, in both cases, $G_{s'}$ passes through Σ^1. By (2.4) and by the fact that $V(s \wedge s')$ is a divisor, it follows that $s' \in H^0(X, \mathcal{F}^{\Sigma}) \setminus \langle s \rangle$.

(ii) Assume that $q \in C \setminus \Sigma$ and that $s'(q) \neq (0,0)$; in this case, after part (i), if (3,17) is equal to 0 at q, for each $1 \leq i \leq 3$, we would have that $(g_1(q), g_2(q))$ is linear dependent on each pair in (3.18). In particular, the three pairs in (3.18) would be linearly dependent. This is a contradiction; indeed, since $q \in C \setminus \Sigma$, the Jacobian map in (2.12) is surjective at q, i.e. $T^1_{C,q} = 0$. On the other hand, since $\mathcal{N}_{C/X,q} \cong \mathcal{O}_{C,q}^{\oplus 2}$, then we must have that two of the three pairs in (3.18) are linearly independent at q.

This implies that $V(s \wedge s')$ cannot be singular outside $\Sigma \cup (C \cap V(s'))$. On the other hand, in the other cases - i.e. either $q \in \Sigma$ or $q \in (C \cap V(s'))$ or both - it is easy to observe that (3.17) always vanishes at q, so that $V(s \wedge s')$ is singular at each such point. □

This shows that the local computations on X for divisors $V(s \wedge s')$, introduced in [2], are equivalent via the (s, G_s)-Severi correspondence to the local computations on $\mathcal{P}$ introduced in Theorem 2.1 and Proposition 2.3.

Furthermore, we remark that surfaces in X given by $V(s \wedge s')$, with $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and $s' \in H^0(X, \mathcal{F}^\Sigma) \setminus \langle s \rangle$, are certainly singular along Σ if the zero-locus $V(s')$ passes there. However, they can be also singular along Σ even if $V(s')$ does not pass there; this happens when each equation on the right-hand side of (3.17) vanishes, since $C = V(s)$ is singular along Σ.

To sum up, we have:

Proposition 3.19

By using notation and assumptions as in Theorem 2.1 and in Proposition 2.3, the following conditions are equivalent:

(i) $s' \in H^0(X, \mathcal{F}^\Sigma) \setminus \langle s \rangle$;

(ii) $V(s \wedge s') \subset X$ *is a surface which contains* C *and which is singular along* Σ;

(iii) the divisor $G_{s'}$ *passes through* Σ^1

(iv) the surface $\mathcal{S}_{s,s'} := G_s \cap G_{s'} \subset \mathcal{P}$ *is singular along* Σ^1;

Proof. Some of the implications are already proven:

(i) $\Leftrightarrow$ *(ii)*: see Proposition 3.14, (i).

(ii) $\Leftrightarrow$ *(iii)*: Let $s' \in H^0(X, \mathcal{F}) \setminus \langle s \rangle$ and let $p \in \Sigma$. Take U an analytical neighbourhood of p and assume that $s'|_U = (g_1, g_2)$, for some $g_1, g_2 \in \mathcal{O}_X(U)$ so that the local equation of $G_{s'}$ in $\pi^{-1}(U)$ is $ug_1 + vg_2 = 0$ (cf. e.g. (3.2)). Two cases can occur. If $s'(p) = 0$, then also $G_{s'}$ contains the fibre L_p and there is nothing to prove. In the other case, $G_{s'}$ passes through the singular point of G_s along L (i.e. $[-\lambda, 1]$ as in (3.8)) if, and only if,

$$[-g_2(p), g_1(p)] = [-\lambda, 1]. \tag{3.20}$$

This means that $[-g_2(p), g_1(p)]$ is a solution of the system (3.5), which is equivalent to the fact that each equation on the right-hand side of (3.17) vanishes; by Proposition 3.14, this is equivalent to the fact that the surface $V(s \wedge s')$ is singular at p.

(iii) $\Leftrightarrow$ *(iv)*: trivial consequence of the fact that G_s is always singular at Σ^1 by Theorem 2.1. □

We observe some differences between the approaches on X and on $\mathcal{P}$ given by the (s, G_s)-Severi correspondence.

Remark 3.21. We observed above that $V(s\wedge s')$, with $s' \in H^0(X, \mathcal{F}^\Sigma)$, is singular along Σ if either $V(s')$ contains it or not. From the correspondence between $V(s\wedge s')$ and $\mathcal{S}_{s,s'}$ we see that in the former case the surface $\mathcal{S}_{s,s'}$ has to contain $\Lambda = \pi^{-1}(\Sigma) = \bigcup_{i=1}^{\delta} L_{p_i}$, whereas in the latter, $\mathcal{S}_{s,s'}$ has to pass through the point $p_i^1 \in L_{p_i}$, $1 \le i \le \delta$, which is singular for G_s so - a fortiori - for $\mathcal{S}_{s,s'}$. In any case, differently from $V(s\wedge s')$, the surface $\mathcal{S}_{s,s'}$ always contains Σ^1 and dominates $V(s\wedge s')$. In particular, if s' is the general section in $H^0(X, \mathcal{F}^\Sigma) \setminus \langle s\rangle$ such that $V(s') \cap C = \emptyset$ then, by the Zariski Main Theorem, $\mathcal{S}_{s,s'}$ is isomorphic via π to the normal surface $V(s\wedge s')$.

Remark 3.22. By using the (s, G_s)-Severi correspondence, one can better understand the fact that the inclusion $\mathcal{J}_{\Sigma/X} \otimes \mathcal{F} \subseteq \mathcal{F}^\Sigma$ is proper, as it follows from

$$0 \to \mathcal{J}_{\Sigma/X} \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_\Sigma \cong \mathcal{O}_\Sigma^{\oplus 2} \to 0 \tag{3.23}$$

and from (2.11). Indeed, by (2.4), the general section $s' \in H^0(X, \mathcal{F}^\Sigma)$ corresponds to a divisor $G_{s'}$ in $|\mathcal{O}_\mathcal{P}(1)|$ which simply passes through the scheme Σ^1 of δ rational double points of the divisor $G_s \in |\mathcal{O}_\mathcal{P}(1)|$. From (3.17), we observe that among elements in $H^0(X, \mathcal{F}^\Sigma)$ there are global sections $s^* \in H^0(X, \mathcal{J}_{\Sigma/X} \otimes \mathcal{F})$. Any of this section determines a divisor $G_{s^*} \in |\mathcal{J}_{\Lambda/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)|$, where $\Lambda = \bigcup_{p_i \in \Sigma} L_{p_i}$. In this case, we have

$$0 \to \mathcal{J}_{\Lambda/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1) \to \mathcal{O}_\mathcal{P}(1) \to \mathcal{O}_\mathcal{P}(1) \otimes \mathcal{O}_\Lambda \cong \bigoplus_{i=1}^{\delta} \mathcal{O}_{L_{p_i}}(1) \to 0, \tag{3.24}$$

where $\Sigma = \{p_1, \dots, p_\delta\}$. Since $\mathcal{O}_{L_{p_i}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, for each $1 \le i \le \delta$, it is clear that $|\mathcal{J}_{\Lambda/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)|$ is properly contained in $|\mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)|$. Therefore, $|\mathcal{J}_{\Lambda/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)|$ has expected codimension equal to 2δ in $|\mathcal{O}_\mathcal{P}(1)|$.

For completeness sake, we conclude by observing that the subsheaf $\mathcal{J}_{C/X} \otimes \mathcal{F} \subset \mathcal{F}^\Sigma$ gives global sections which are related in the (s, G_s)-Severi correspondence to divisors in $|\mathcal{J}_{\mathbb{F}/\mathcal{P}} \otimes \mathcal{O}_\mathcal{P}(1)|$, where $\mathbb{F} := Proj(Sym(\mathcal{F}|_C)) = \mathbb{P}_C^1$ is a singular, ruled surface contained in $\mathcal{P}$ with π-fibres over the base curve C.

Remark 3.25. To conclude this section, we observe that Propositions 2.3 and 3.19 give several distinct but equivalent characterizations of tangent vectors to $\mathcal{V}_\delta(\mathcal{F})$ at $[s]$. These conditions are consistent with those of regularity in Corollary 2.15.

Recall that the map $\rho_\mathcal{P}$ in (2.16) is a standard restriction map. Therefore, $|\mathcal{O}_\mathcal{P}(1)|$ does not separate Σ^1 if, and only if, each divisor in $|\mathcal{O}_\mathcal{P}(1)|$ passing through all but one point p_j^1 of Σ^1 passes also through the point p_j^1, for some $1 \le j \le \delta$. By Theorems 2.1 and 3.19 and by Proposition 2.3, this happens if, and only if, for each $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and for each $s' \in H^0(X, \mathcal{F}^\Sigma)$, the surface $\mathcal{S}_{s,s'} = G_s \cap G_{s'}$ which is singular along all but one point p_j^1 of Σ^1 is singular also at the remaining point p_j^1, for some $1 \le j \le \delta$. This happens if, and only if, for each $[s] \in \mathcal{V}_\delta(\mathcal{F})$ and for each $s' \in H^0(X, \mathcal{F}^\Sigma)$, the surface $V(s\wedge s') \subset X$ which is singular along all but one point p_j of Σ is singular also at the remaining point p_j, for some $1 \le j \le \delta$; this is equivalent to the non-surjectivity of the map μ_X, since the section $s' \in H^0(X, \mathcal{F})$ which vanishes in the composition $\mathcal{F} \twoheadrightarrow \mathcal{F}|_C \twoheadrightarrow \mathcal{O}_{\Sigma \setminus \{p_j\}}$ also vanishes in the composition $\mathcal{F} \twoheadrightarrow \mathcal{F}|_C \twoheadrightarrow \mathcal{O}_{\{p_j\}}$.

4. $\mathcal{P}_\delta(\mathcal{F})$-Severi varieties of singular divisors on $\mathcal{P}$

What showed up to now suggests that the equivalences given by Theorems 2.1, 3.1 for geometric singular loci, by Propositions 2.3, 3.19 for tangent vectors and by Corollary 2.15 for vector space maps reside in a more deep geometric equivalence of families of singular objects. Indeed, as we shall prove in Corollary 4.19 and Theorem 5.1, the procedure introduced in [12] and recalled in § 2 rephrases Severi varieties of nodal sections $\mathcal{V}_\delta(\mathcal{F})$ on X in terms of "classical" Severi varieties of some singular divisors on smooth ruled fourfolds. This, once more, motivates the terminology introduced in Definition 2.2 and highlights the rich geometry which is behind the (s, G_s)-Severi correspondence.

By taking into account Theorem 3.1 and Remark 3.12, we give the following:

DEFINITION 4.1. With notation as in Assumption 1 and Theorem 2.1, consider the scheme

$$\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) := \{G_s \in |\mathcal{O}_\mathcal{P}(1)| \text{ s.t. } [s] \in \mathcal{V}_\delta(\mathcal{F})\}. \tag{4.2}$$

For any $\mathcal{F}$ and δ, these schemes parametrize families of divisors in the tautological linear system $|\mathcal{O}_\mathcal{P}(1)|$ which are irreducible, with δ rational double points as the only singularities and which are related to elements in $\mathcal{V}_\delta(\mathcal{F})$. For brevity sake, these will be called $\mathcal{P}_\delta(\mathcal{F})$-*Severi varieties*.

From now on, we shall always consider $\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1)) \neq \emptyset$. It is clear that:

$$\operatorname{expdim}(\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))) = \dim(|\mathcal{O}_\mathcal{P}(1)|) - \delta; \tag{4.3}$$

indeed, imposing a rational double point gives at most 5 conditions on $|\mathcal{O}_\mathcal{P}(1)|$; each such point varies on any of the π-fibre over X.

As in Definition 1.5, from (4.3) it is natural to give the following:

DEFINITION 4.4. Let $[G_s] \in \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))$. Then $[G_s]$ is said to be a *regular point* of $\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))$ if:

(i) $[G_s] \in \mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))$ is a smooth point, and
(ii) $\dim_{[G_s]}(\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))) = \operatorname{expdim}(\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))) = \dim(|\mathcal{O}_\mathcal{P}(1)|) - \delta$.

The $\mathcal{P}_\delta(\mathcal{F})$-Severi variety $\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))$ is said to be *regular* if it is regular at each point.

As in Theorem 2.1, in order to find regularity conditions it is crucial to give a description of the tangent space at a point $[G_s]$ to the given $\mathcal{P}_\delta(\mathcal{F})$-Severi variety $\mathcal{R}_\delta(\mathcal{O}_\mathcal{P}(1))$.

Theorem 4.5

Let $[G_s] \in \mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ on $\mathcal{P}$ and let Σ^1 be the zero-dimensional scheme of the δ-rational double points of $G_s \subset \mathcal{P}$. Then:

$$T_{[G_s]}\big(\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))\big) \cong \frac{H^0(\mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1))}{\langle G_s \rangle}. \tag{4.6}$$

In particular, if $\epsilon \in \mathbb{C}[T]/(T^2)$ is such that $\epsilon^2 = 0$, then:

$$G_s + \epsilon\, G_r \in T_{[G_s]}(\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1)) \Leftrightarrow G_r \in |\mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1))|.$$

Proof. The divisor $G_s \subset \mathcal{P}$, related to the point $[G_s] \in \mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$, corresponds to a section $[s] \in \mathcal{V}_\delta(\mathcal{F})$ on X. Therefore, if $p \in \Sigma = Sing(C)$ is a node and if $U \subset X$ is an analytical neighbourhood containing p, then the local equation of G_s in $\pi^{-1}(U) \cong U \times \mathbb{P}^1$ is given by $ux_1x_2 + vx_3 = 0$ (cf. formula (3.9) and the proof of Theorem 3.4 (i) in [12]).

In the open chart where $v \neq 0$, G_s is smooth whereas, in the open chart where $u \neq 0$, the local equation of G_s is

$$G_s = V(x_1x_2 + x_3t), \tag{4.7}$$

where $t = \frac{v}{u}$ and (x_1, x_2, x_3, t) coordinates in $\mathbb{A}^4$.

We can consider the Jacobian map of G_s in this $\mathbb{A}^4$. This is given by:

$$\begin{array}{rcl} \mathcal{T}_{\mathbb{A}^4|_{G_s}} & \xrightarrow{J_{G_s}} & \mathcal{N}_{G_s/\mathbb{A}^4} \\ \partial/\partial x_1 & \longrightarrow & x_2 \\ \partial/\partial x_2 & \longrightarrow & x_1 \\ \partial/\partial x_3 & \longrightarrow & t \\ \partial/\partial t & \longrightarrow & x_3, \end{array}$$

where $\mathcal{N}_{G_s/\mathbb{A}^4}$ is locally free of rank one on G_s. It is then clear that J_{G_s} is surjective except at the origin $\underline{0} \in \mathbb{A}^4$, i.e. at the singularity of G_s in U. By local analytical computations, we get:

$$coker(J_{G_s}) \cong \frac{\mathbb{C}[[x_1, x_2, x_3, t]]/(x_1x_2 + x_3t)}{(x_1, x_2, x_3, t)} \cong \mathbb{C}; \tag{4.8}$$

Globally speaking, given $G_s \subset \mathcal{P}$ whose singular scheme is Σ^1, we have the exact sequence of sheaves on G_s:

$$\mathcal{T}_{\mathcal{P}|_{G_s}} \xrightarrow{J_{G_s}} \mathcal{N}_{G_s/\mathcal{P}} \to T^1_{G_s} \to 0, \tag{4.9}$$

where $T^1_{G_s}$ is a sky-scraper sheaf supported on Σ^1 and of rank one at each point of Σ^1 by (4.8).

As in (2.9) for nodal curves, denote by $\mathcal{N}'_{G_s}$ the image of J_{G_s} in (4.9). This is the *equisingular sheaf*, whose global sections parametrize equisingular first-order deformations of G_s in $\mathcal{P}$. From (4.8) and from the fact that $\mathcal{N}_{G_s/\mathcal{P}}$ is locally free of rank-one, it follows that

$$\mathcal{N}'_{G_s} \cong \underline{m}_{p^1}, \text{ for all } p^1 \in \Sigma^1, \tag{4.10}$$

where $\underline{m}_{p^1} \subset \mathcal{O}_{G_s,p^1}$ is the maximal ideal of $p^1 \in G_s$.

On the other hand, one can consider the standard diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{O}_{\mathcal{P}} & \stackrel{\cdot G_s}{\to} & \mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1) & \to & \mathcal{J}_{\Sigma^1/G_s} \otimes \mathcal{O}_{G_s}(1) & \to 0 \\
 & || & & \downarrow & & \downarrow & \\
0 \to & \mathcal{O}_{\mathcal{P}} & \stackrel{\cdot G_s}{\to} & \mathcal{O}_{\mathcal{P}}(1) & \to & \mathcal{O}_{G_s}(1) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \to & \mathcal{O}_{\Sigma^1} & \stackrel{\cong}{\to} & \mathcal{O}_{\Sigma^1} & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array} \tag{4.11}$$

by (4.10) and (4.11), we get that there is an injection

$$\frac{H^0(\mathcal{P}, \mathcal{J}_{\Sigma^1/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}}(1))}{H^0(\mathcal{P}, \mathcal{O}_{\mathcal{P}})} \hookrightarrow H^0(G_s, \mathcal{N}'_{G_s}), \tag{4.12}$$

which is an isomorphism when $\mathcal{P}$ (equivalently X) is regular, i.e. $h^1(\mathcal{P}, \mathcal{O}_{\mathcal{P}}) = 0$. Therefore, the vector space on the left-hand-side of (4.12) actually parametrizes equisingular first-order deformations of G_s in $|\mathcal{O}_{\mathcal{P}}(1)|$. $\square$

Remark 4.13. Recall that when one studies classical Severi varieties of irreducible, δ-nodal curves on a smooth projective surface, there is also a *parametric approach* for equisingular first-order deformations (cf., e.g [4] and [22]).

Precisely, let S be an arbitrary smooth, projective surface, $|\mathcal{O}_S(D)|$ a complete linear system on S, whose general element is assumed to be a smooth and irreducible curve. One considers the Severi variety $V_{|\mathcal{O}_S(D)|,\delta}$, for any $\delta \leq p_a(D)$, which parametrizes reduced, irreducible curves in $|\mathcal{O}_S(D)|$ having δ-nodes as the only singularities. If $[C] \in V_{|\mathcal{O}_S(D)|,\delta}$, this point corresponds to a curve $C \sim D$ on S, such that $N := Sing(C) \subset S$ is the 0-dimensional scheme of its δ nodes; one can consider:

$$\begin{array}{ccc}
\tilde{C} & \subset & \tilde{S} \\
\downarrow {\scriptstyle \varphi_N} & & \downarrow {\scriptstyle \mu_N} \\
C & \subset & S
\end{array} \tag{4.14}$$

where

- μ_N is the blow-up of S along N,
- φ_N is the normalization of C,
- $\tilde{C}$ is a smooth, irreducible curve of (geometric) genus $g = g(\tilde{C}) = p_a(D) - \delta$.

It is a standard result that $T_{[C]}(V_{|\mathcal{O}_S(D)|,\,\delta}) \cong \frac{H^0(S,\,\mathcal{J}_{N/S}(D))}{\langle C \rangle}$ is isomorphic to a (proper) subspace of $H^0(\mathcal{N}_{\varphi_N})$, where $\mathcal{N}_{\varphi_N}$ is the *normal bundle to the map* φ_N which is the line bundle on $\tilde{C}$ defined by:

$$0 \to \mathcal{T}_{\tilde{C}} \to \varphi^*(\mathcal{T}_S) \to \mathcal{N}_{\varphi_N} \to 0.$$

It is well-known that $H^0(\tilde{C}, \mathcal{N}_{\varphi_N})$ parametrizes equisingular first-order deformations of C in S and that the subspace $T_{[C]}(V_{|\mathcal{O}_S(D)|,\,\delta})$, parametrizing equisingular first-order deformations of C in $|\mathcal{O}_S(D)|$, coincides with the whole vector space when e.g. S is a regular surface.

On the other hand, for irreducible nodal curves on surfaces, the parametric approach coincides with the *Cartesian approach*, which makes use of the equisingular sheaf $\mathcal{N}'_C$ defined as in (2.9). Indeed, in the surface case, one has $\mathcal{N}'_C \cong \varphi_*(\mathcal{N}_{\varphi_N})$ (cf. e.g. [22]).

Therefore, if in particular S is e.g. regular one has

$$T_{[C]}(V_{|\mathcal{O}_S(D)|,\,\delta}) \cong H^0(C, \mathcal{N}'_C) \cong H^0(\tilde{C}, \mathcal{N}_{\varphi_N}). \tag{4.15}$$

The same does not occur for divisors in $\mathcal{P}$ which are elements of $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$, even if we assume for simplicity that X (and so $\mathcal{P}$) is regular, i.e. $h^1(X, \mathcal{O}_X) = 0$.

Indeed, let

$$\mu_{\Sigma^1} : \tilde{\mathcal{P}} \to \mathcal{P} \ \text{ and } \ \varphi_{\Sigma^1} : \tilde{G}_s \to G_s$$

be the blow-up of $\mathcal{P}$ along Σ^1 and the desingularization of G_s, respectively. The map φ_{Σ^1} is induced by μ_{Σ^1} as it follows by using a diagram similar to (4.14) and by the fact that Σ^1 is a scheme of ordinary double points for G_s. Let $B := \Sigma_{i=1}^{\delta} E_i$ be the μ_{Σ^1}-exceptional divisor. Thus,

$$\mu^*_{\Sigma^1}(G_s) = \tilde{G}_s + 2B, \ \ \mu^*_{\Sigma^1}(K_{\mathcal{P}}) = K_{\tilde{\mathcal{P}}} - 3B.$$

By the exact sequence:

$$0 \to \mathcal{T}_{\tilde{G}_s} \to \varphi^*_{\Sigma^1}(\mathcal{T}_{\mathcal{P}}) \to \mathcal{N}_{\varphi_{\Sigma^1}} \to 0$$

and by the adjunction formula on $\tilde{\mathcal{P}}$, we get that:

$$\mathcal{N}_{\varphi_{\Sigma^1}} \cong \mathcal{O}_{\tilde{G}_s}(\mu^*_{\Sigma^1}(G_s) + B). \tag{4.16}$$

Tensoring by $\mathcal{O}_{\tilde{G}_s}(\mu^*_{\Sigma^1}(G_s) + B)$ the exact sequence

$$0 \to \mathcal{O}_{\tilde{\mathcal{P}}}(-\mu^*_{\Sigma^1}(G_s) + 2B) \to \mathcal{O}_{\tilde{\mathcal{P}}} \to \mathcal{O}_{\tilde{G}_s} \to 0,$$

we get

$$0 \to \mathcal{O}_{\tilde{\mathcal{P}}}(3B) \to \mathcal{O}_{\tilde{\mathcal{P}}}(\mu^*_{\Sigma^1}(G_s) + B) \to \mathcal{N}_{\varphi_{\Sigma^1}} \to 0.$$

By Fujita's Lemma (see e.g. [18], Lemma 1-3-2) and by the fact that B is effective and μ_{Σ^1}-exceptional, we get:

$$H^0(\mathcal{O}_{\tilde{\mathcal{P}}}(\mu^*_{\Sigma^1}(G_s) + B)) \cong H^0(\mathcal{O}_{\mathcal{P}}(1)) \text{ and } H^i(\mathcal{O}_{\tilde{\mathcal{P}}}(3B)) \cong H^i(\mathcal{O}_{\mathcal{P}}),\ \forall\, i \geq 0.$$

By the regularity of X, we have $H^1(\mathcal{P}, \mathcal{O}_{\mathcal{P}}) = (0)$. Therefore, we have

$$H^0(\mathcal{N}_{\varphi_{\Sigma^1}}) \cong \frac{H^0(\mathcal{P}, \mathcal{O}_{\mathcal{P}}(1))}{H^0(\mathcal{P}, \mathcal{O}_{\mathcal{P}})} = H^0(G_s, \mathcal{O}_{G_s}(1)). \tag{4.17}$$

On the other hand, from the regularity assumption of X, (4.12) is an isomorphism, so $H^0(\mathcal{N}'_{G_s}) \cong H^0(\mathcal{J}_{\Sigma^1/G_s} \otimes \mathcal{O}_{G_s}(1))$. Thus, by (4.6) and by (4.17), differently from (4.15) in this case we have:

$$T_{[G_s]}\big(\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))\big) \cong H^0(G_s, \mathcal{N}'_{G_s}) \subset H^0(\tilde{G}_s, \mathcal{N}_{\varphi_{\Sigma^1}}). \tag{4.18}$$

In particular, first-order deformations given by general vectors in $H^0(\mathcal{N}_{\varphi_{\Sigma^1}})$ are not equisingular.

To conclude the section, let $\rho_{\mathcal{P}}$ be as in (2.14). Then, from Theorem 4.5, it immediately follows:

Corollary 4.19

With assumptions and notation as in Theorem 4.5, we have:

$$\begin{aligned} [G_s] \in \mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1)) \text{ is a regular point} \Leftrightarrow\ & \rho_{\mathcal{P}} \text{ is surjective} \\ \Leftrightarrow\ & [s] \in \mathcal{V}_\delta(\mathcal{F}) \text{ is a regular point} \\ & \text{(in the sense of Definition 1.5).} \end{aligned}$$

Proof. The first equivalence is a direct consequence of (4.3) and Theorem 4.5. The other follows from Corollary 2.15. □

5. Some uniform regularity results for $\mathcal{V}_\delta(\mathcal{F})$ and $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$

In this section we deduce regularity results for $\mathcal{P}_\delta(\mathcal{F})$-Severi varieties $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ of irreducible divisors in $|\mathcal{O}_{\mathcal{P}}(1)|$ on $\mathcal{P}$. By using a similar approach for regularity results in [12] for Severi varieties $\mathcal{V}_\delta(\mathcal{F})$ on X, we find upper-bounds on the number δ which ensure the regularity of $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$; these upper-bounds are shown to be almost sharp (cf. Remark 5.3).

This approach highlights once more the importance of the (s, G_S)-Severi correspondence. Indeed, if one considers the $\mathcal{P}_\delta(\mathcal{F})$-Severi varieties independently from the corresponding varieties $\mathcal{V}_\delta(\mathcal{F})$ on X, the regularity condition for a point of $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$

is equivalent to the separation of suitable zero-dimensional schemes by the linear system $|\mathcal{O}_{\mathcal{P}}(1)|$ on the fourfold $\mathcal{P}$ (cf. Corollary 4.19). In general, it is well-known how difficult is to establish when a linear system separates points in projective varieties of dimension greater than or equal to three (cf. e.g. [1], [9] and [17]). In some cases, some separation results can be found by using technical tools like *multiplier ideals* as well as the Nadel and the Kawamata-Viehweg vanishing theorems (see, e.g. [8], for an overview). In our situation, thanks to the correspondence between $\mathcal{V}_\delta(\mathcal{F})$ on X and $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ on $\mathcal{P}$, we deduce regularity conditions for $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ from those already obtained for $\mathcal{V}_\delta(\mathcal{F})$ in [12].

From now on, let X be a smooth projective threefold, $\mathcal{E}$ a globally generated rank-two vector bundle and M a very ample line bundle on X; let $k \geq 0$ and $\delta > 0$ be integers. With notation and assumptions as in Section 1, we shall always take

$$\mathcal{F} = \mathcal{E} \otimes M^{\otimes k}$$

and consider the scheme $\mathcal{V}_\delta(\mathcal{E} \otimes M^{\otimes k})$ on X.

One can use for $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ the same approach of Theorem 4.5 in [12], where we considered the more particular case of $X \subset \mathbb{P}^r$ and $M = \mathcal{O}_X(1)$ and where we determined conditions on $\mathcal{E}$ and on k and uniform upper-bounds on the number of nodes δ implying that each point of $\mathcal{V}_\delta(\mathcal{E} \otimes M^{\otimes k})$ is regular.

Theorem 5.1

Let X be a smooth projective threefold, $\mathcal{E}$ a globally generated rank-two vector bundle on X, M a very ample line bundle on X and $k \geq 0$ and $\delta > 0$ integers.
Let $\mathcal{P} := \mathbb{P}_X(\mathcal{E} \otimes M^{\otimes k})$ and let $\mathcal{O}_{\mathcal{P}}(1)$ be its tautological line bundle. Let $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ be the $\mathcal{P}_\delta(\mathcal{F})$-Severi variety of irreducible divisors having δ-rational double points on $\mathcal{P}$. Then, if:

$$\delta \leq k + 1, \tag{5.2}$$

$\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ is regular.

Proof. One observes that (5.2) is a sufficient condition for the regularity of $\mathcal{V}_\delta(\mathcal{E} \otimes M^{\otimes k})$ on X; the proof is analogous to that of Theorem 4.5 in [12], where the case $X \subset \mathbb{P}^r$ and $M = \mathcal{O}_X(1)$ has been considered. Then, one can conclude by using Corollary 4.19. □

Remark 5.3. Observe that the bound (5.2) is uniform, i.e. it does not depend on the postulation of either the rational double points of divisors in $\mathcal{R}_\delta(\mathcal{O}_{\mathcal{P}}(1))$ or the nodes of the curves which are zero-loci of sections parametrized by $\mathcal{V}_\delta(\mathcal{E} \otimes M^{\otimes k})$.

Furthermore, in [12] we observed that the bound $\delta \leq k+1$ is effective and almost sharp. Indeed, as introduced in [2] for the asymptotic case, one can easily construct examples of non-regular points $[s] \in \mathcal{V}_{k+4}(\mathcal{O}_{\mathbb{P}^3}(k+1) \oplus \mathcal{O}_{\mathbb{P}^3}(k+4))$, for any $k \geq 3$, whose corresponding curve C has its $(k+4)$ nodes lying on a line $L \subset \mathbb{P}^3$; anyhow, one can also show that $\mathcal{V}_{k+4}(\mathcal{O}_{\mathbb{P}^3}(k+1) \oplus \mathcal{O}_{\mathbb{P}^3}(k+4))$ is generically regular.

References

1. U. Angehrn and Y.-T. Siu, Effective freeness and point separation for adjoint bundles, *Invent. Math.* **122** (1995), 291–308.
2. E. Ballico and L. Chiantini, A look into the Severi varieties of curves in higher codimension. [Dedicated to the memory of Fernando Serrano], *Collect. Math.* **49** (1998), 191–201.
3. W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, Springer, Berlin, 1984.
4. L. Caporaso and J. Harris, Counting plane curves of any genus, *Invent. Math.* **131** (1998), 345–392.
5. L. Chiantini and C. Ciliberto, On the Severi varieties on surfaces in $\mathbb{P}^3$, *J. Algebraic Geom.* **8** (1999), 67–83.
6. L. Chiantini and A.F. Lopez, Focal loci of families and the genus of curves on surfaces, *Proc. Amer. Math. Soc.* **127** (1999), 3451–3459.
7. L. Chiantini and E. Sernesi, Nodal curves on surfaces of general type, *Math. Ann.* **307** (1997), 41–56.
8. L. Ein, Multiplier ideals, vanishing theorems and applications, *Proc. Sympos. Pure Math.* **62** (1997), 203–219.
9. L. Ein and R. Lazarsfeld, Global generation of pluricanonical and adjoint linear series on smooth projective threefolds, *Amer. Math. Soc.* **6** (1993), 875–903.
10. F. Flamini, *Families of nodal curves on projective surfaces*, Ph.D thesis, Consortium Universities of Rome "La Sapienza" and "Roma Tre", (2000). Available on http://www.mat.uniroma3.it/dipartimento/esterni/flamin$i_$ricerca.html
11. F. Flamini, Moduli of nodal curves on smooth surfaces of general type, *J. Algebraic Geom.* **11** (2002), 725–760.
12. F. Flamini, Families of nodal curves on projective threefolds and their regularity via postulation of nodes, *to be published on Trans. Amer. Math. Soc.* (2003). Elettronically published on *Trans. Amer. Math. Soc.* S 0002 - 9947(03)03199-4. The preprint version *math.AG/0202022, 4 Feb. 2002* is slightly stronger than the published one.
13. R. Friedman, *Algebraic Surfaces and Holomorphic Vector Bundles* (UTX), Springer-Verlag, New York, 1998.
14. G.M. Greuel, C. Lossen, and E. Shustin, Geometry of families of nodal curves on the blown-up projective plane, *Trans. Amer. Math. Soc.* **350** (1998), 251–274.
15. J. Harris, On the Severi problem, *Invent. Math.* **84** (1986), 445–461.
16. R. Hartshorne, *Algebraic Geometry* (GTM No. 52), Springer-Verlag, New York - Heidelberg, 1977.
17. Y. Kawamata, On Fujita's freeness conjecture for 3-folds and 4-folds, *Math. Ann.* **308** (1997), 491–505.
18. Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model problem, in *Algebraic Geometry, Sendai, Adv. Stud. Pure Math.* **10**, North-Holland, Amsterdam, 1987, 283–360.
19. C. Okonek, M. Schneider, and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Progress in Mathematics **3**, Boston-Basel-Stuttgart, Birkhäuser, 1980.
20. S. Lichtenbaum and M. Schlessinger, The cotangent complex of a morphism, *Trans. Amer. Math. Soc.* **128** (1967), 41–70.
21. Z. Ran, On nodal plane curves, *Invent. Math.* **86** (1986), 529–534.
22. E. Sernesi, On the existence of certain families of curves, *Invent. Math.* **75** (1984), 25–57.
23. F. Severi, *Vorlesungen Über Algebraische Geometrie*, Teubner, Leipzig, 1921.
24. L. Szpiro, *Lectures on Equations Defining Space Curves*, Tata Institute of Fundamental Research, Bombay, Springer, 1979.