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A note on a theorem of Xiao Gang

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Abstract

In 1985 Xiao Gang proved that the bicanonical system of a complex surface S of general type with $p_2(S) > 2$ is not composed of a pencil ([8]). In this note a new proof of this theorem is presented.

1. Introduction

In 1985, in [8], Xiao Gang proved that the bicanonical image of a (minimal) surface of general type S is not a surface (i.e., the bicanonical system is composed of a pencil) if and only if $K_S^2 = 1$, $p_g(S) = 0$, i.e., if and only if $p_2(S) = 2$.

When, in the end of the 80's, it was finally proven that $|2K_S|$ is base point free, whenever $p_g \geq 1$, the part of this theorem concerning surfaces with $p_g \geq 1$ became trivial.

The aim of this note is to give a brief new proof of this theorem of Xiao Gang, using several results which by now are standard techniques of surface theory.

Notations and conventions. A *surface* is an algebraic projective surface over $\mathbb{C}$. No distinction is made between line bundles and divisors on a smooth variety, and additive and multiplicative notation are used interchangeably. Linear equivalence is denoted by $\equiv$ and numerical equivalence by $\sim$. Given a linear system $|D|$ on a surface, the corresponding rational map is denoted by φ_D, and $|D|$ is *composed of a pencil* if dim Im $\varphi_D = 1$. The remaining notation is standard in algebraic geometry.

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2. Preliminaries

Here we list the results which will be needed in the sequel.

Proposition 2.1 ([9], Théorème 2.2)

Let S be a minimal surface of general type with $p_g(S) = 0$. If S has a genus 2 fibration $f\colon S \to \mathbb{P}^1$, then $K_S^2 \leq 2$.

We will need also:

Proposition 2.2 ([1], Corollaire 5.8)

A minimal surface of general type S satisfying $q(S) = 0$ and $K_S^2 \leq 2\chi(\mathcal{O}_S)$ has no irregular étale covers.

Theorem 2.3 (De Franchis, [6], cf. [3], [7])

Let S be a smooth surface with $p_g(S) = q(S) = 0$ and $\pi\colon Y \to S$ a smooth double cover with $q(Y) > 0$. Then:

(i) the Albanese map of Y is a fibration $\alpha\colon Y \to B$ over a curve $B \subset \mathrm{Alb}\, S$;
(ii) there exist a fibration $g\colon S \to \mathbb{P}^1$ and a degree 2 map $p\colon B \to \mathbb{P}^1$ such that $p \circ \alpha = g \circ \pi$.

Finally we recall:

Theorem 2.4

The bicanonical system $|2K_S|$ of a minimal surface of general type S is base point free, if $p_g(S) \geq 1$ or $K_S^2 \geq 5$.

This result is due to the work of several authors. For the corresponding references see the survey paper [5].

3. The proof of Xiao Gang's theorem

The theorem will follow from Theorem 2.4 and the following:

Theorem 3.1

Let S be a minimal surface of general type with $p_g = 0$. Then $|2K_S|$ is not composed of a pencil if and only if $K_S^2 > 1$.

Proof. Since $\chi(\mathcal{O}_S)=1$, one has $h^0(S,2K_S)=K_S^2+1$ and therefore, if $K_S^2=1$, $|2K_S|$ is a pencil.

Suppose now that $K_S^2>1$ and that $|2K_S|$ is composed of a pencil with general fibre F. Since $q=0$, $|F|$ is a rational pencil and so $2K_S\equiv dF+Z$, where $d=K_S^2$ and Z is an effective divisor possibly zero.

Now, because K_S is nef, we have $2K_S^2\equiv dK_SF+K_SZ\geq dK_SF$ and therefore $K_SF\leq 2$. The index theorem yields $K_S^2F^2\leq (K_SF)^2$ where if equality holds then for some $a,b\in\mathbb{Q}$, $aK_S\sim bF$. Also by the adjunction formula K_SF and F^2 have the same parity. By the assumption $K_S^2>1$, one obtains the following the numerical possibilities:

i) $K_SF=2$, $F^2=2$ and $K_S\sim F$;

ii) $K_SF=2$, $F^2=0$.

Now case i) does not occur. Suppose otherwise. Then $2K_S\equiv 2F$ and so $\eta:=K_S-F$ is a 2-torsion divisor. The étale double cover $p:Y\to S$ associated to η satisfies $\chi(\mathcal{O}_Y)=2$ and $p_g(Y)=h^0(S,K_S)+h^0(S,K_S+\eta)=h^0(S,K_S)+h^0(S,F)=2$. So Y is irregular. This is a contradiction to Proposition 2.2, and case i) is excluded.

For case ii) notice that, anyway, $K_S^2=2$, because in this case the pencil $|F|$ is a genus 2 fibration and therefore $K_S^2\leq 2$, by Proposition 2.1.

Then one has $2K_S\equiv 2F+Z$, where $Z>0$ is such that $K_SZ=0$, and so every irreducible curve in Z is a $-2-$curve. The effective divisor Z can be decomposed as $Z=2Z_0+Z_1$ where Z_0,Z_1 are effective divisors, and Z_1 is reduced.

If $Z_1=0$, then $\eta:=K_S-F-Z_0$ is a 2-torsion divisor and the same argument as above leads again to a contradiction.

Suppose now that $Z_1\neq 0$. Since $Z_1=2(K_S-F-Z_0)$, θZ_1 is even for any irreducible component θ of Z_1. On the other hand the dual graph of the configuration of curves in Z_1 is a union of trees and thus, because Z_1 is reduced, necessarily Z_1 is a disjoint union of p irreducible $-2-$curves. So Z_1 is a smooth divisor and we can consider the double cover $p:Y'\to S$ branched on Z_1 and defined by the relation $Z_1=2(K_S-F-Z_0)$.

The standard double cover formulas yield

$$\begin{aligned}\chi(\mathcal{O}_{Y'})&=2\chi(\mathcal{O}_S)+\frac{1}{2}(K_S-F-Z_0)(2K_S-F-Z_0)=2-\frac{p}{4};\\ K_{Y'}^2&=2(2K_S-F-Z_0)^2=4-p;\\ p_g(Y')&=h^0(S,\mathcal{O}_S(2K_S-F-Z_0))+h^0(S,\mathcal{O}_S(K_S))=2.\end{aligned}$$

Since the surface Y' is of course of general type, the only possibility is that $p=4$ ($\chi(\mathcal{O}_{Y'})=1$). Now $p_g(Y')=2$ yields $q(Y')=2$ and thus, by De Franchis Theorem 2.3, Y' is not of Albanese general type.

On the other hand, the minimal model Y of Y' is obtained by contracting the four exceptional curves which are the inverse images of the four components of Z_1 and as such satisfies $K_Y^2=4$. If we denote by f the genus of a general fibre of the Albanese pencil of Y, we have then a contradiction to Arakelov's inequality $K_Y^2\geq 8(f-1)(q(Y)-1)$ (see, e.g., [2]).

So also this case does not occur and the theorem is proven. □

Remark. Very recently Meng Chen and E.Viehweg also found a different proof of Xiao's theorem, which uses vanishing theorems for $\mathbb{Q}-$divisors (see [4]).

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