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A regularity result for p-harmonic equations with measure data

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Abstract

We examine the p-harmonic equation div $|\nabla u|^{p-2}\nabla u = \mu$ where μ is a bounded Radon measure. We determine a range of p's for which solutions to the equation verify an a priori estimate. For such p's we also prove an higher integrability result.

1. Introduction

The paper is concerned with the non homogeneous p-harmonic equation

$$\operatorname{div} |\nabla u|^{p-2}\nabla u \;=\; \operatorname{div} f \qquad \text{on} \qquad \Omega \tag{1.1}$$

where Ω is a bounded open regular subset of $\mathbb{R}^n$, $n \geq 2$. When u is a function in the Sobolev class $W^{1,p}_o(\Omega)$ and $f = (f^1, \ldots, f^n)$ is a vector field in $L^q(\Omega;\mathbb{R}^n)$, with $\frac{1}{p} + \frac{1}{q} = 1$, we are in the so-called "natural setting" of the p-harmonic equation. A function u is referred to as a solution of equation (1.1) if the distributional gradient of u verifies the integral identity

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega f\nabla\varphi \tag{1.2}$$

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for every $\varphi \in C_o^\infty(\Omega)$. Of course, if $u \in W_o^{1,p}(\Omega)$, by an approximation argument, (1.2) extends to all $\varphi \in W_o^{1,p}(\Omega)$ as well. Then we can apply (1.2) to $\varphi = u$ and immediately obtain the following basic estimate

$$\int_\Omega |\nabla u|^p \le \int_\Omega |f|^q .$$

Although $L^q(\Omega)$ is the natural space to which the vector field f has to belong, many recent papers have been devoted to the study of the p-harmonic equation (1.1) when the right hand side belongs to a space different from L^q.

This study began with a paper by Iwaniec and Sbordone ([14]), where the p-harmonic equation is examined for $f \in L^{q\pm\varepsilon}(\Omega;\mathbb{R}^n)$. They proved that if $u \in W^{1,p\pm\varepsilon}(\Omega)$ is a solution to the equation (1.1) for a suitable small $\varepsilon > 0$, then $u \in W^{1,p}(\Omega)$.

Analogous regularity results have been established later on for more general type of operators that are power-like in ∇u (see [8], [5]). A motivation for the study of (1.1) when the right hand side is the divergence of a vector field belonging to a space different from the natural one can be found in the equation

$$\operatorname{div} |\nabla u|^{p-2}\nabla u \,=\, \mu \qquad \text{on} \qquad \Omega \tag{1.3}$$

where μ is a Radon measure of finite total variation in Ω.

Namely, such a measure μ can be written as $\mu = \operatorname{div} F$, with some $F \in L^s(\Omega,\mathbb{R}^n)$ and $s = \frac{n-\epsilon}{n-1}$, for every $\epsilon > 0$ (see Lemma 3.4 below and the recent paper [3]).

Properties of the distributional solutions to equation (1.3) have been investigated only when $p = n$. In that case, in [7, 10] is proved that there exists a unique distributional solution which belongs to the grand Sobolev space $W^{1,n)}(\Omega)$, i.e. the space of functions u such that

$$||\nabla u||_{L^{n)}} = \sup_{0<\epsilon\le n-1} \left(\epsilon \fint_\Omega |\nabla u|^{n-\epsilon}\right)^{1/(n-\epsilon)} < \infty .$$

There are of course many more possible spaces in which the equation (1.3) admits solutions, in case $p \le n$. Such spaces lay beyond the range of our paper. For them we refer to [1], [2], [6], [12], [17].

As remarked in [15], for investigating properties of the distributional solutions to the equation

$$\operatorname{div} |\nabla u|^{p-2}\nabla u \,=\, \mu = \operatorname{div} F \qquad F \in L^{(n-\epsilon)/(n-1)}(\Omega,\mathbb{R}^n)$$

what we need first are estimates of the form

$$\int_\Omega |\nabla u|^{((p-1)(n-\epsilon))/(n-1)} \le C(n,p) \int_\Omega |F|^{(n-\epsilon)/(n-1)} \tag{1.4}$$

that are known only in case $p = n$ (see [7], [10]). In this paper we fix a bounded Radon measure μ and determine a range of p's for which solutions of the p-harmonic equation (1.3) satisfy estimate (1.4). Namely, we have the following.

Theorem 1

Let $f \in L^r(\Omega)$, $r > 1$. There exists $\delta = \delta(n, r) > 0$ such that if $p > \max\{\frac{r}{r-1} - \delta;$ $1 + \frac{1}{r}\}$ and $g \in L^{r(p-1)}(\Omega)$ then every solution $u \in W_o^{1,r(p-1)}(\Omega)$ of the equation

$$\operatorname{div}\left(|\nabla u + g|^{p-2}(\nabla u + g)\right) = \operatorname{div} f \tag{1.5}$$

satisfies the following estimate

$$||\nabla u||^{p-1}_{L^{r(p-1)}(\Omega)} \leq C||f||_{L^r(\Omega)} + C||g||^{p-1}_{L^{r(p-1)}(\Omega)} . \tag{1.6}$$

Moreover for every $r > 1$ there exists $\delta = \delta(n, r) > 0$ such that if $|p-2| < \delta$ then every solution $u \in W_o^{1,r(p-1)}(\Omega)$ of the equation (1.5) satisfies estimate (1.6).

Having estimate (1.6) at our disposal, we establish that a solution $u \in W_o^{1,r(p-1)}$ of equation (1.5) satisfies a reverse Hölder inequality, from which we get the following higher integrability result.

Theorem 2

Let $f \in L^{r+\eta}(\Omega)$, $r > 1$, $\eta > 0$. If p is related to r as in Theorem 1 and $u \in W_o^{1,r(p-1)}(\Omega)$ is a solution of the equation (1.1), then $u \in W_{loc}^{1,r(p-1)+\sigma}(\Omega)$, some $\sigma = \sigma(r, n, \eta) > 0$.

Our results can be rewritten for the distributional solutions to the equation (1.3) where μ is a bounded Radon measure.

Theorem 3

Let μ be a bounded Radon measure on Ω. There exists $\delta > 0$ such that if $p < n$ and one of the two following conditions holds

i) $n - \delta < p$

ii) $|p - 2| < \delta$

a solution $u \in W_o^{1,\frac{s(p-1)}{n-1}}(\Omega)$, with $\frac{s(p-1)}{n-1} > 1$, actually belongs to $W_{loc}^{1,\frac{r(p-1)}{n-1}}(\Omega)$, for any $s < r < n$.

Remark 1.1. A slight modification of the arguments presented here shows that our results remain valid for the $\mathcal{A}$-harmonic operator

$$\operatorname{div}\mathcal{A}(x, \nabla u)$$

where $\mathcal{A}$ satisfies the usual growth and coercivity conditions.

Remark 1.2. We could obtain similar results also when the ellipticity bounds are not L^∞ but belong to the space BMO of functions of bounded mean oscillation, just using the arguments developed in [4], [15], [11]. More precisely, let us consider the equation

$$\operatorname{div}(b(x)|\nabla u|^{p-2}\nabla u) = \operatorname{div} f \tag{1.7}$$

where $1 \leq b(x)$ is a function in the space $BMO(\Omega)$ and $f \in L^r(\Omega)$. There exists $\delta = \delta(n, r, ||b||_{BMO}) > 0$ such that if $p > \max\{\frac{r}{r-1} - \delta; 1 + \frac{1}{r}\}$ or $|p-2| < \delta$ then every solution $u \in W_o^{1,r(p-1)}(\Omega)$ of the equation (1.7) satisfies the following estimate

$$||\nabla u||^{p-1}_{L^{r(p-1)}(\Omega)} \leq C||f||_{L^r(\Omega)} .$$

To avoid technicalities, we present the proof in the simplest case.

2. Preliminary results

In order to get a priori estimate for the solution to p-Laplace equation one usually tests the identity (1.2) with functions φ such that $\nabla\varphi$ are essentially proportional to ∇u. Unfortunately, when $p < n$, u cannot be used as test function in our problem. We have to construct admissible test functions and then we need the following.

Theorem 2.1 (Hodge decomposition)

Let w belong to $W_o^{1,s}(\Omega)$, with $s > 1$ and let $-1 < \varepsilon < s - 1$. Then there exist $\phi \in W_o^{1,\frac{s}{1+\varepsilon}}(\Omega)$ and a divergence free vector field $H \in L^{\frac{s}{1+\varepsilon}}(\Omega)$ such that

$$|\nabla w|^{\varepsilon}\nabla w = \nabla\phi + H. \tag{2.1}$$

Moreover

$$||\nabla\phi||_{L^{\frac{s}{1+\varepsilon}}} \leq C_1||\nabla w||^{1+\varepsilon}_{L^s} \tag{2.2}$$

$$||H||_{L^{\frac{s}{1+\varepsilon}}} \leq C_2(n, s)|\varepsilon|||\nabla w||^{1+\varepsilon}_{L^s} . \tag{2.3}$$

Proof. See Theorem 3 in [14]. □

In order to obtain a reverse Hölder inequality for the solutions of our equation, we shall use the following Poincaré - Sobolev Lemma.

Lemma 2.2

Let $B(x_o, R)$ be a ball in $\mathbb{R}^n$ and $A \in L^1_{loc}(B; \mathbb{R}^{m\times n})$ be a matrix field. There exists a divergence free matrix field $A_B \in L^1_{loc}(B; \mathbb{R}^{m\times n})$ such that

$$\left(\fint_B |A(x) - A_B|^r\right)^{1/r} \leq C(r, m, n)R\left(\fint_B |\text{div}\, A|^s\right)^{1/s}$$

provided $s \geq \max\{1, \frac{nr}{n+r}\}$ and $\text{div}\, A \in L^s(B; \mathbb{R}^m)$.

Proof. See Lemma 6.1 in [14]. □

The higher integrability result follows by applying the well-known.

Theorem 2.3

Let $h \in L^1(\Omega)$ and suppose that for concentric balls $\frac{B}{2} = B(x_o, \frac{R}{2}) \subset B = B(x_o, R) \subset \Omega$ we have

$$\fint_{B/2} h(x)dx \leq C \left(\fint_B h(x)^m dx \right)^{1/m} + \fint_B k(x)dx$$

some $0 < m < 1$. If $k \in L^t(\Omega)$, with $t > 1$ then there exists an exponent $r > 1$ such that $h \in L^r(\frac{B}{2})$ and

$$\fint_{B/2} h^r(x)dx \leq C \left(\fint_B h(x)dx \right)^r + \fint_B k^r(x)dx\,.$$

Proof. See [9]. □

As we have already mentioned, each bounded Radon measure on Ω, can be written as the divergence of a suitable vector field F belonging to the space $L^{\frac{n}{n-1})}(\Omega, \mathbb{R}^n)$. Namely, we have:

Lemma 3.4

Given a bounded Radon measure μ on Ω, there exists a vector field F such that

$$\mathrm{div} F = \mu$$

and

$$||F||_{L^{\frac{n}{n-1})}} = \sup_{0<\epsilon\leq\frac{1}{n-1}} \left(\epsilon \fint_\Omega |F|^{(n-\epsilon)/(n-1)} \right)^{(n-1)/(n-\epsilon)} \leq C \int_\Omega |d\mu|\,.$$

Proof. See [7], [10]. □

3. The a priori estimate

In this section we give the proof of Theorem 1. We confine ourselves to the case $\frac{p}{p-1} > r$. When $\frac{p}{p-1} < r$, estimate of Theorem 1 has been proved in [13].

Proof of Theorem 1. Hodge decomposition stated in Theorem 2.1 implies that there exist $\varphi \in W_o^{1,\frac{r}{r-1}}(\Omega)$ and a divergence free vector field $H \in L^{\frac{r}{r-1}}(\Omega; \mathbb{R}^n)$ such that

$$|\nabla u|^{r(p-1)-p} \nabla u = \nabla\varphi + H$$

and

$$||\nabla\varphi||_{L^{\frac{r}{r-1}}} \leq C_1 ||\nabla u||^{r(p-1)-p+1}_{L^{r(p-1)}} \tag{3.1}$$

$$||H||_{L^{\frac{r}{r-1}}} \leq C_2(n,r)|r(p-1)-p| ||\nabla u||^{r(p-1)-p+1}_{L^{r(p-1)}} . \tag{3.2}$$

Note that φ is an admissible test function in equation (1.5). Therefore using the following Lipschitz property of the p-laplacian

$$\big| |a+b|^{p-2}(a+b) - |a|^{p-2}a \big| \leq c(p)|b|\big(|a|+|b|\big)^{p-2} \qquad \forall a,b \in \mathbb{R}^n$$

and that u is a solution, we get

$$\begin{aligned}
\int_\Omega |\nabla u|^{r(p-1)} &= \int_\Omega \langle |\nabla u|^{p-2}\nabla u, |\nabla u|^{r(p-1)-p}\nabla u\rangle \\
&= \int_\Omega \langle |\nabla u|^{p-2}\nabla u - |\nabla u+g|^{p-2}(\nabla u+g), |\nabla u|^{r(p-1)-p}\nabla u\rangle \\
&\quad + \int_\Omega \langle |\nabla u+g|^{p-2}(\nabla u+g), |\nabla u|^{r(p-1)-p}\nabla u\rangle \\
&\leq c(p)\int_\Omega |g|(|\nabla u|+|\nabla u+g|)^{p-2}|\nabla u|^{r(p-1)-p+1} \\
&\quad + \int_\Omega \langle f, \nabla\varphi\rangle + \int_\Omega \langle |\nabla u+g|^{p-2}(\nabla u+g), H\rangle .
\end{aligned}$$

Hölder inequality and estimate (3.1) imply

$$\begin{aligned}
\int_\Omega |\nabla u|^{r(p-1)} &\leq c\int_\Omega |g|(|\nabla u|+|\nabla u+g|)^{r(p-1)-1} \\
&\quad + ||f||_{L^r}||\nabla\varphi||_{L^{\frac{r}{r-1}}} + \int_\Omega \langle |\nabla u+g|^{p-2}(\nabla u+g), H\rangle \\
&\leq C||g||_{L^r(p-1)}|||\nabla u|+|\nabla u+g|||^{r(p-1)-1}_{L^r(p-1)} \\
&\quad + C_1||f||_{L^r}||\nabla u||^{r(p-1)-p+1}_{L^{r(p-1)}} + \int_\Omega \langle |\nabla u+g|^{p-2}(\nabla u+g), H\rangle . \qquad (3.4)
\end{aligned}$$

Using Hodge decomposition again, we have

$$|\nabla u+g|^{p-2}(\nabla u+g) = \nabla\psi + K \tag{3.5}$$

where $\psi \in W^{1,r}_0(\Omega)$, div K=0 and

$$||K||_{L^r} \leq \tilde{C}_2(n,r)|p-2|||\nabla u + g||^{p-1}_{L^{r(p-1)}} \,. \tag{3.6}$$

From this estimate, recalling that H is divergence free and using (3.2) and (3.6) we get

$$\begin{aligned}
\int_\Omega \langle |\nabla u + g|^{p-2}(\nabla u + g), H\rangle &= \int_\Omega \langle \nabla\psi, H\rangle + \int_\Omega \langle K, H\rangle \\
&= \int_\Omega \langle K, H\rangle \leq ||H||_{L^{\frac{r}{r-1}}} ||K||_{L^r} \leq C_2\tilde{C}_2|r(p-1) \\
&\quad - p||p-2|(||\nabla u||^{r(p-1)}_{L^{r(p-1)}} + ||g||^{r(p-1)}_{L^{r(p-1)}}) \,.
\end{aligned} \tag{3.7}$$

Inserting (3.7) in (3.4) and using Young's inequality, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^{r(p-1)} &\leq C_1||f||_{L^r}||\nabla u||^{r(p-1)-p+1}_{L^{r(p-1)}} \\
&\quad + C_2\tilde{C}_2|r(p-1)-p||p-2|(||\nabla u||^{r(p-1)}_{L^{r(p-1)}} + ||g||^{r(p-1)}_{L^{r(p-1)}}) \,.
\end{aligned} \tag{3.8}$$

From this inequality if r, p are such that

$$C_2\tilde{C}_2|r(p-1)-p||p-2| < 1 \tag{3.9}$$

holds, we immediately get

$$||\nabla u||^{r(p-1)}_{L^{r(p-1)}} \leq C||f||^r_{L^r} + C||g||^{r(p-1)}_{L^{r(p-1)}}$$

as claimed. □

4. Higher integrability

Throughout this section the exponents p and r are related as in Theorem 1 and $u \in W^{1,r(p-1)}_o(\Omega)$ is a solution of equation (1.5), with $g = 0$.

Proof of Theorem 2. Fix a function $\varphi \in C^\infty_0(\Omega)$ and introduce the function $w = \varphi u$. Routine calculations show that w, belonging to the space $W^{1,r(p-1)}_o(\Omega)$, solves the equation

$$\text{div}|\nabla w|^{p-2}\nabla w = \text{div}G$$

where

$$G = \varphi^{p-1}|\nabla u|^{p-2}\nabla u + |\varphi\nabla u + u\nabla\varphi|^{p-2}(\varphi\nabla u + u\nabla\varphi) - |\varphi\nabla u|^{p-2}(\varphi\nabla u) \,.$$

Since

$$|G| \leq |\varphi|^{p-1}|\nabla u|^{p-1} + |u\nabla\varphi|(|u\nabla\varphi| + |\varphi\nabla u|)^{p-2} \in L^r(\Omega)$$

we are legitimate to apply estimate (1.6) to the function w and find

$$||\nabla w||^{p-1}_{r(p-1)} \le C||G||_r \le C||\varphi^{p-1}|\nabla u|||^{p-1}_{r(p-1)} + C||u\nabla\varphi||^{p-1}_{r(p-1)}\,. \tag{4.1}$$

Now, fix a ball $B(x_o,R) \subset \Omega$ and let $\varphi \in C^\infty_0(B)$ be a cut-off function such that $0 \le \varphi \le 1$, $\varphi = 1$ on $B(x_o,\frac{R}{2})$ and $|\nabla\varphi| \le \frac{c}{R}$. Writing inequality (4.1), using the properties of φ and estimate (1.6) for the function u, we obtain

$$⨍_{B(x_o,R/2)} |\nabla u|^{r(p-1)} \le \frac{C}{R} ⨍_{B(x_o,R)} |u|^{r(p-1)} + C ⨍_{B(x_o,R)} |f|^{r(p-1)}\,.$$

Using Sobolev-Poincaré inequality we get to the following reverse Hölder inequality

$$⨍_{B(x_o,R/2)} |\nabla u|^{r(p-1)} \le C\left(⨍_{B(x_o,R)} |\nabla u|^s\right)^{(r(p-1))/s} + ⨍_{B(x_o,R)} |f|^r$$

provided $s \ge \frac{nr(p-1)}{n+r(p-1)}$. The result follows by applying Theorem 2.3. □

Proof of Theorem 3. Using Lemma 3.4 we can express μ as the divergence of a vector field f belonging to $L^s(\Omega)$, for every $s < \frac{n}{n-1}$. Then we are legitimate to apply Theorem 2 to find, for p verifying

$$C(n)|s(p-1)-p||p-2| < 1, \tag{4.2}$$

that a solution $u \in W_o^{1,\frac{s(p-1)}{n-1}}(\Omega)$ actually belongs to $W_{loc}^{1,\frac{s(p-1)}{n-1}+\eta}(\Omega)$ some $\eta > 0$. We conclude the proof iterating this process, since the range of p's found via inequality (4.2) has positive Lebesgue measure when s tends to $\frac{n}{n-1}$. □

References

1. P. Bènilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), 241–273.
2. L. Boccardo and T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.* **87** (1989), 149–169.
3. J. Bourgain and H. Brezis, On the equation $div\, Y = f$ and application to control of phases, *J. Amer. Math. Soc.* **16** (2003), 393–426.
4. M. Carozza, G. Moscariello, and A. Passarelli di Napoli, Nonlinear equations with growth coefficients in BMO, *Houston J. Math.* **28** (2002), 917–929.
5. M. Carozza and A. Passarelli di Napoli, On very weak solutions of a class of nonlinear elliptic systems, *Comment. Math. Univ. Carolin.* **41** (2000), 493–508.
6. G. Dolzmann, N. Hungerbühler, and S. Müller, Uniqueness and maximal regularity for nonlinear elliptic systems of n-Laplace type with measure valued right hand side, *J. Reine Angew. Math.* **520** (2000), 1–35.
7. A. Fiorenza and C. Sbordone, Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1, *Studia Math.* **127** (1998), 223–231.

8. D. Giachetti and R. Schianchi, Boundary higher integrability for the gradient of distributional solutions of nonlinear systems, *Studia Math.* **123** (1997), 175–184.
9. E. Giusti, *Metodi Diretti nel Calcolo delle Variazioni*, U.M.I. Bologna, 1994.
10. L. Greco, T. Iwaniec, and C. Sbordone, Inverting the p-harmonic operator, *Manuscripta Math.* **92** (1997), 249–258.
11. L. Greco and A. Verde, A regularity property of p-harmonic functions, *Ann. Acad. Sci. Fenn. Math.* **25** (2000), 317–323.
12. T. Kilpeläinen and J. Malỳ, Degenerate elliptic equations with measure data and nonlinear potentials, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **19** (1992), 591–613.
13. T. Iwaniec, Projections onto gradient fields and L^p-estimates for degenerated elliptic operators, *Studia Math.* **75** (1983), 293–312.
14. T. Iwaniec and C. Sbordone, Weak minima of variational integrals, *J. Reine Angew. Math.* **454** (1994), 143–161.
15. T. Iwaniec and C. Sbordone, Quasiharmonic fields, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **18** (2001), 519–572.
16. F. Murat, Equations nonlinéaires avec second membre in L^1 ou mésure, Preprint 1994.
17. G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble)* **15** (1965), 189–258.